BONFERRONI TYPE TESTS FOR RETURN PREDICTABILITY WITH POSSIBLY TRENDING PREDICTORS

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Outline of Presentation

Table of Contents

- We develop tests for predictability that are robust to the degree of persistence of the predictor and can be validly applied regardless of whether the predictor admits a deterministic linear trend or only requires a simple mean-correction.
- While the popular Bonferroni Q test of Campbell and Yogo (2006) [CY] displays excellent power properties for strongly persistent predictors that admit only a deterministic level, we show that it can suffer from severe size distortions and power losses when the predictor is either trending, or is weakly persistent (or both).

- We relax the assumptions of CY by allowing for the possible presence of a (local) deterministic linear trend in the predictor series.
- We derive the limiting distribution of the Bonferroni Q test of CY and the Bonferroni t test of Cavanagh *et al.* (1995) [CES] when a local linear trend is present in the predictor. These results highlight that these tests are asymptotically undersized (oversized) when testing in the right (left) tail when the innovations to the predictor and returns are negatively correlated.
- In response we develop trend-augmented versions of the Bonferroni Q test and Bonferroni t test are developed that are invariant to the magnitude of any trend present (so it need not be local-to-zero) in the predictor series.

Motivation - Right Tailed Testing

- For right tailed testing, the resulting trend-augmented tests are exact invariant to the magnitude of the linear trend term and display superior size control and power to the asymptotically undersized mean-only Bonferroni Q and Bonferroni t tests when a trend is present in the predictor.
- On the other hand, when no trend is present (its magnitude is exactly zero) in the predictor, the trend-augmented tests are less powerful than their mean-only counterparts.
- In practice uncertainty will exist of whether a trend is present in the predictor or not. Accordingly, we propose a union-of-rejections approach based on both the mean-only and trend-augmented Bonferroni Q tests. This strategy able to capture the superior power of the mean-only test when no trend is present, and that of the trend-augmented test for trending predictors.
- We further refine this approach by switching into the trend augmented Bonferroni t test or standard t test as evidence of strong persistence in the predictor weakens.

Motivation - Left Tailed Testing

- For left tailed testing we find asymptotic oversize in the mean-only Bonferroni Q and Bonferroni t tests. An implication of this is that a union of rejections strategy is not feasible as the asymptotic size of the procedure cannot be controlled.
- For left tailed testing we therefore initially recommend employing the trend-augmented Bonferroni Q test for strongly persistent predictors.
- Again, we refine this approach by switching into the trend-augmented Bonferroni t test or standard t test as evidence of strong persistence in the predictor weakens.

Table of Contents

We consider the following predictive regression model

$$r_t = \alpha + \beta(x_{t-1} - \gamma(t-1)) + u_t, \qquad t = 2, ..., T \tag{1}$$

where r_t denotes the return on an asset in period t, and x_{t-1} denotes a putative predictor observed at time $t-1$. We assume the process for x_t is given by

$$\begin{aligned} x_t &= \mu + \gamma t + w_t, \qquad t = 1, ..., T & (2) \\ w_t &= \rho w_{t-1} + v_t, \qquad t = 2, ..., T & (3) \end{aligned}$$

where w_1 is assumed to be an $O_p(1)$ random variable and where u_t and v_t are disturbances.

Assumption 1

We assume that $\psi(L)v_t = e_t$ *where* $\psi(L) := \sum_{i=0}^{p-1} \psi_i L^i$ *with* $\psi_0 = 1$ *and* $\psi(1) \neq 0$, *with the roots of* $\psi(L)$ *assumed to be less than one in absolute value.*

We assume that $z_t := (u_t, e_t)'$ *is a bivariate martingale difference sequence with respect to the natural filtration* $\mathcal{F}_t := \sigma\{z_s, s \leq t\}$ *satisfying the following conditions: (i)* $E[z_t z_t'] = \begin{bmatrix} \sigma_u^2 & \sigma_{ue} \\ \sigma_{ue} & \sigma_e^2 \end{bmatrix}$, *(ii)* $\sup_t E[u_t^4] < \infty$, *and (iii)* $\sup_t E[e_t^4] < \infty$.

We define $\omega_v^2 := \lim_{T\to\infty} T^{-1} E(\sum_{t=2}^{T} v_t)^2 = \sigma_e^2/\psi(1)^2$ *to be the long run variance of the error process* $\{v_t\}$, *and* $\delta := \sigma_{ue}/\sigma_u\sigma_e$ *as the correlation between the innovations* $\{u_t\}$ *and* $\{e_t\}$.

Assumptions for x_t and γ

Assumption S

The predictor $\{x_t\}$ is strongly persistent, with the autoregressive parameter ρ in (3) given by $\rho = \rho_T = 1 + cT^{-1}$ with c a finite non-zero constant.

Assumption W

The predictor $\{x_t\}$ is weakly persistent, with the autoregressive parameter ρ in (3) fixed and bounded away from unity, $|\rho| < 1$.

Assumption T

The trend coefficient γ in (1) and (2) is given by $\gamma = \gamma_T = \kappa\omega_v T^{-1/2}$, where κ is a finite constant.

The Bonferroni Q test

- The Bonferroni Q test of CY is constructed under the assumption that Assumption S holds.
- To perform this test, an initial $100(1-\alpha_1)\%$ (asymptotic) confidence interval for ρ is calculated by inverting some mean-only unit root test statistic, with this confidence interval denoted $[\underline{\rho}, \bar{\rho}]$.
- An equal tailed $100(1-\alpha_2)\%$ confidence interval for β given ρ is obtained by regressing $r_t - \hat{\sigma}_{ue}(\hat{\sigma}_e\hat{\omega}_v)^{-1}(x_t - \underline{\rho}x_{t-1})$ and $r_t - \hat{\sigma}_{ue}(\hat{\sigma}_e\hat{\omega}_v)^{-1}(x_t - \bar{\rho}x_{t-1})$, respectively, on a constant and x_{t-1}.
- By Bonferroni's inequality this CI for β will have coverage of at least $100(1-\alpha)\%$ where $\alpha := \alpha_1 + \alpha_2$.

The Refined Bonferroni Q test

- CY find this method can be very be conservative so, for a given value of δ, they propose a refined method where the value of α_1 is chosen to give one-sided tests for predictability with maximum asymptotic size of 5% when Assumption S holds.
- We omit details for the Bonferroni t test of CES, but a similar refined Bonferroni strategy is employed to deliver a test with maximum asymptotic size of 5% when Assumption S holds.
- Following CY and CES we utilise the constant only ADF-GLS and ADF-OLS unit root test statistics when performing the original Bonferroni Q and t tests, respectively.
- Henceforth we refer to these mean-only tests as Q_μ^{GLS} and t_μ^{OLS}

Limit Distribution of Mean-Only Test Statistics Under Assumption S

Theorem 1

Let data be generated according to (1)-(3). Let $W_1(s)$ and $W_2(s)$ be independent standard Brownian motion processes and let $W_{1c}(r) = \int_0^r e^{(r-s)c} dW_1(s)$. Define $\tilde{\rho} := 1 + \tilde{c}T^{-1}$. Then under Assumptions 1, S and T, and under the local alternative $H_b : \beta = \beta_T = b(\sigma_u/\omega_v)T^{-1}$,

$$(a) \qquad t_\mu \xrightarrow{w} \frac{b\left\{\kappa \int_0^1 rW_{1c}^{\mu}(r)dr + \int_0^1 W_{1c}^{\mu}(r)^2 dr\right\} + \delta \int_0^1 W_{1c}^{\mu,\kappa}(r)dW_1(r)}{\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}} + \sqrt{1-\delta^2} Z_\mu$$

$$(b) \qquad Q_\mu(\tilde{\rho}) \xrightarrow{w} \frac{b\left[\kappa \int_0^1 rW_{1c}^{\mu}(r)dr + \int_0^1 W_{1c}^{\mu}(r)^2 dr\right] + \delta c \kappa \int_0^1 rW_{1c}^{\mu,\kappa}(r)dr + \delta(\tilde{c}-c)\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}{\sqrt{1-\delta^2}\sqrt{\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr}} + Z_\mu$$

where $\xrightarrow{w}$ denotes weak convergence of the associated probability measures, and where $W_{1c}^{\mu}(r) := W_{1c}(r) - \int_0^1 W_{1c}(s)ds$, $W_{1c}^{\mu,\kappa}(r) := \{\kappa(r-0.5)\} + W_{1c}^{\mu}(r)$ and $Z_\mu := \left(\int_0^1 W_{1c}^{\mu,\kappa}(r)^2 dr\right)^{-1/2} \int_0^1 W_{1c}^{\mu,\kappa}(r)dW_2(r)$

- The limiting distribution of $Q_\mu(\tilde{\rho})$ is a function of the value of $\tilde{\rho} = 1 + \tilde{c}/T$. In practice with c unknown, with this value of $\tilde{\rho}$ will be obtained from an initial confidence interval for ρ, constructed by inverting the mean-only ADF-GLS unit root test statistic.
- Observe that when $\kappa = 0$, such that no trend is present in the predictor, these distributions simplify to those obtained by CES and CY.
- The local asymptotic power of the CES and CY tests will also depend on the limit distribution of the unit root test statistics which are used to obtain the initial confidence intervals for ρ, these are given by

$$DF\text{-}OLS_\mu \xrightarrow{w} \frac{(\kappa/2 + W_{1c}^{\mu}(1))^2 - (-\kappa/2 + W_{1c}^{\mu}(0))^2 - 1}{2\sqrt{\int_0^1 \{\kappa(r - 1/2) + W_{1c}^{\mu}(r)\}^2 \, dr}}$$

$$DF\text{-}GLS_\mu \xrightarrow{w} \frac{(\kappa + W_{1c}(1))^2 - 1}{2\sqrt{\int_0^1 \{\kappa r + W_{1c}(r)\}^2 \, dr}}$$

Table of Contents

- We initially develop trend-augmented versions of both the Bonferroni Q and t tests, denoting these as Q_{τ}^{GLS} and t_{τ}^{OLS}, respectively.
- These tests are performed in an identical manner to the extant Q_{μ}^{GLS} and t_{μ}^{OLS} tests, with the difference being that a linear trend is included in the estimated predictive regressions and the initial confidence interval for ρ is obtained from a trend-augmented unit root test statistic. For Q_{τ}^{GLS} we use the with-trend ADF-GLS unit root test statistic and for t_{τ}^{OLS} we use the ADF-OLS unit root test statistic.
- Following CY we again select the significance level used to construct the initial confidence interval for ρ, α_1, such that for a given value of δ the Q_{τ}^{GLS} and t_{τ}^{OLS} tests have maximum asymptotic size of 5% in each tail over a grid of values of $c \in [-50, 5]$.
- We will later discuss our hybrid test procedures whose aim is to exploit the superior power of Q_{μ}^{GLS} when $\kappa = 0$ and Q_{τ}^{GLS} when $\kappa \neq 0$.

Theorem 2

Let data be generated according to (1)-(3). Let $W_1(s)$ and $W_2(s)$ be independent standard Brownian motion processes and let $W_{1c}(r) = \int_0^r e^{(r-s)c} dW_1(s)$. Define $\tilde{\rho} := 1 + \tilde{c}T^{-1}$. Then under Assumptions 1, S and T, and under the local alternative $H_b : \beta = \beta_T = b(\sigma_u/\omega_v)T^{-1}$,

$$(a) \quad t_\tau \xrightarrow{w} b\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr} + \delta \frac{\int_0^1 W_{1c}^\tau(r) dW_1(r)}{\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}} + \sqrt{1-\delta^2} Z_\tau$$

$$(b) \quad Q_\tau(\tilde{\rho}) \xrightarrow{w} \frac{[b + \delta(\tilde{c} - c)]\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}}{\sqrt{1-\delta^2}} + Z_\tau.$$

where $W_{1c}^\tau(r) := W_{1c}^\mu(r) - 12(r-0.5)\int_0^1 (s-0.5) W_{1c}(s) ds$ and $Z_\tau := \left(\int_0^1 W_{1c}^\tau(r)^2 dr\right)^{-1/2} \int_0^1 W_{1c}^\tau(r) dW_2(r)$.

Limit Distribution of Trend Augmented Test Statistics Under Assumption S

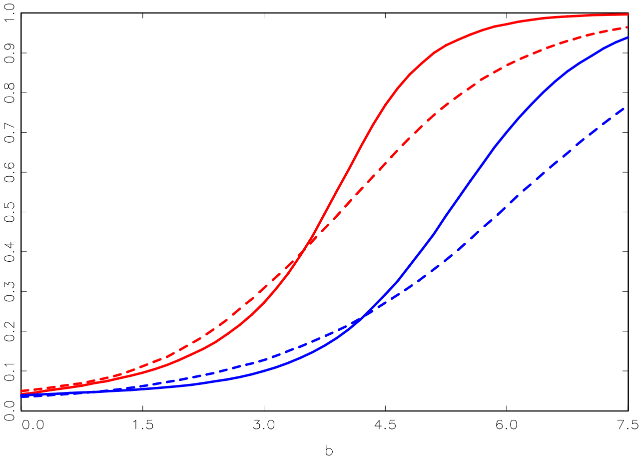
- The limiting distribution of $Q_\tau(\tilde{\rho})$ is, again, a function of the value of $\tilde{\rho} = 1 + \tilde{c}/T$. In practice, again, this value will be obtained from the initial confidence interval for ρ constructed by inverting the with-trend ADF-GLS unit root test statistic.
- The local asymptotic power of the trend-augmented CY and CES tests again depend on the limit distribution of the relevant unit root test statistic used to obtain the initial confidence interval for ρ, these are given by

$$DF\text{-}OLS_\tau \xrightarrow{w} \frac{W_{1c}^\tau(1)^2 - W_{1c}^\tau(0)^2 - 1}{2\sqrt{\int_0^1 W_{1c}^\tau(r)^2 dr}}$$

$$DF\text{-}GLS_\tau^{\overline{c}} \xrightarrow{w} \frac{W_{1c}^{\tau,\overline{c}}(1)^2 - 1}{2\sqrt{\int_0^1 W_{1c}^{\tau,\overline{c}}(r)^2 dr}}$$

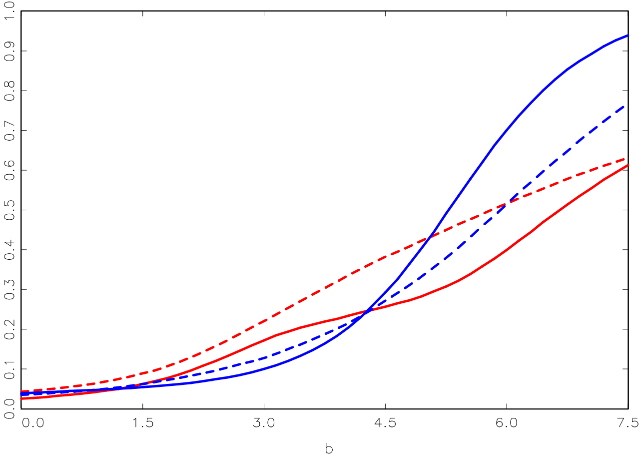
where $W_{1c}^{\tau,\overline{c}}(r) := W_{1c}(r) - r\left\{\overline{c}^* W_{1c}(1) + 3(1-\overline{c}^*)\int_0^1 rW_{1c}(r)dr\right\}$ and $\overline{c}^* := (1-\overline{c})/(1-\overline{c}+\overline{c}^2/3)$, where $\bar{c}$ is the pseudo-GLS de-trending parameter (usually $\bar{c} = -13.5$).

Local Asymptotic Power. $c = 0$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



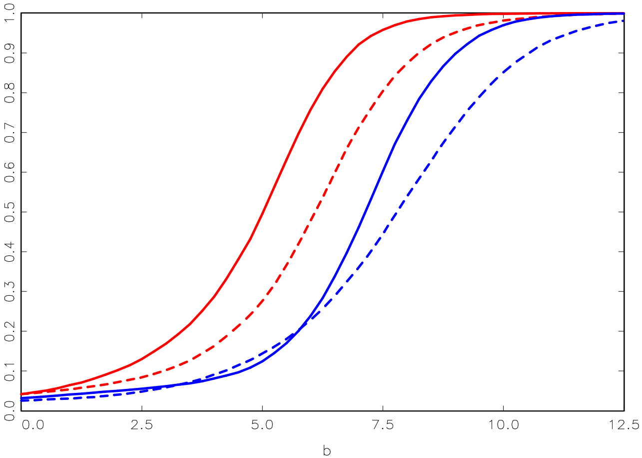
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = 0$, $\delta = -0.95$, $\kappa = 1.0$. Right Tail.



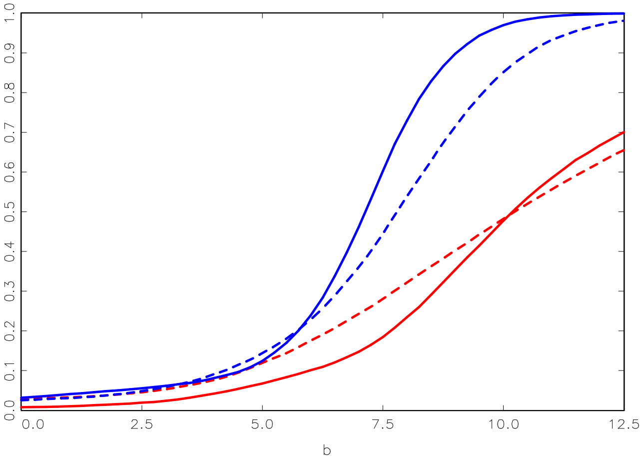
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -2$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



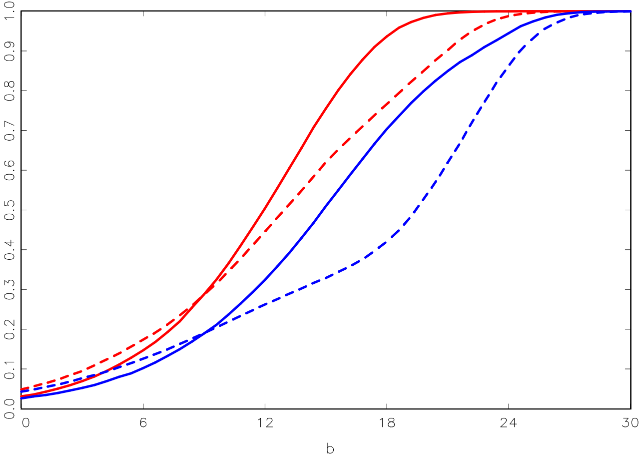
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -2$, $\delta = -0.95$, $\kappa = 1.0$. Right Tail.



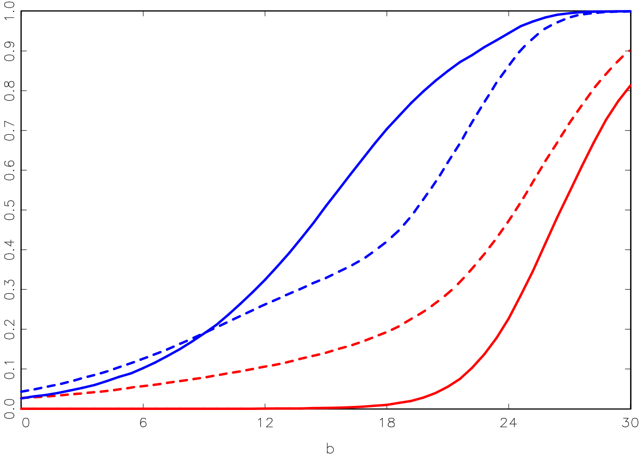
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -20$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



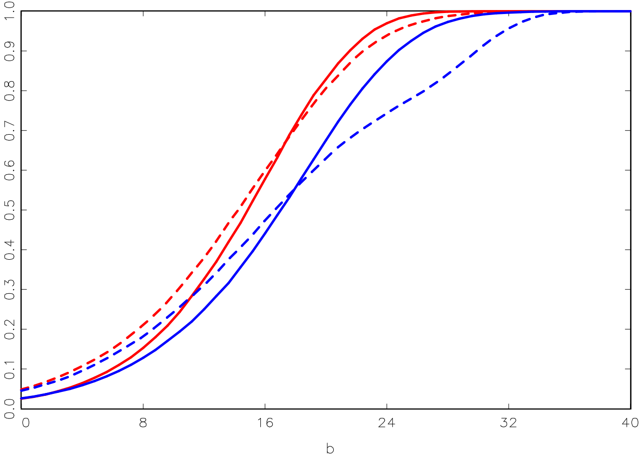
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -20$, $\delta = -0.95$, $\kappa = 0.5$. Right Tail.

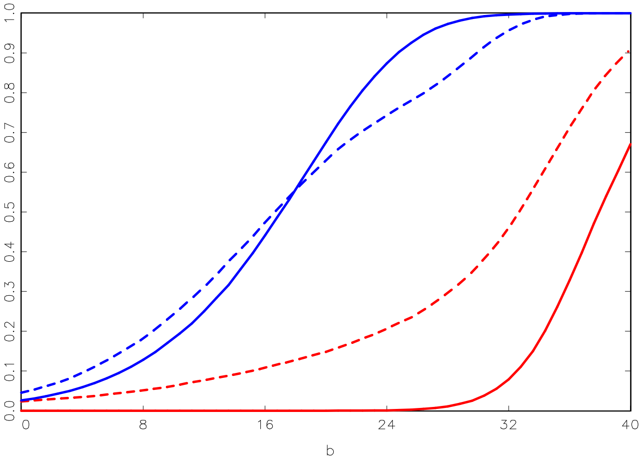


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -30$, $\delta = -0.95$, $\kappa = 0$. Right Tail.

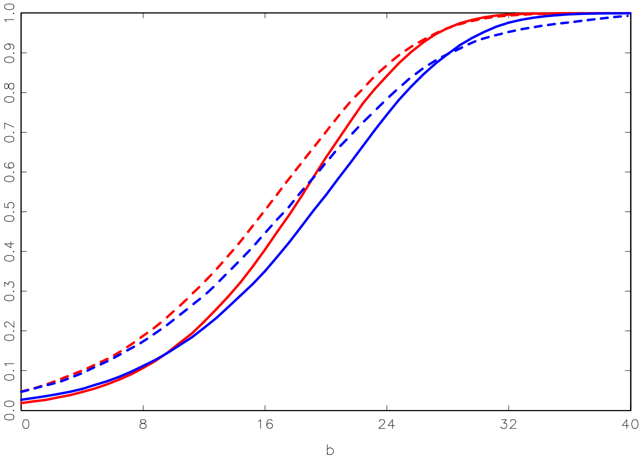


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——



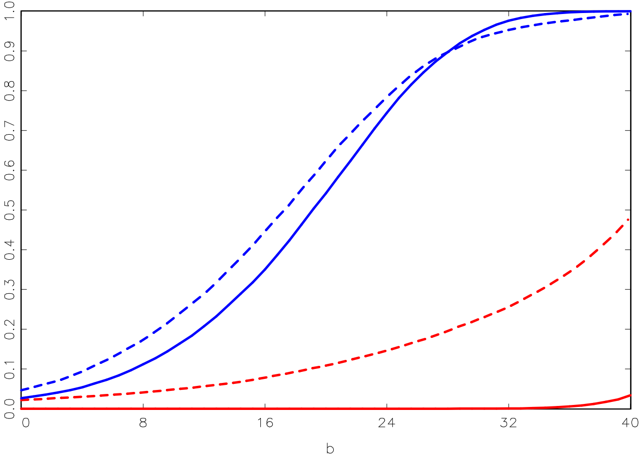
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -40$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



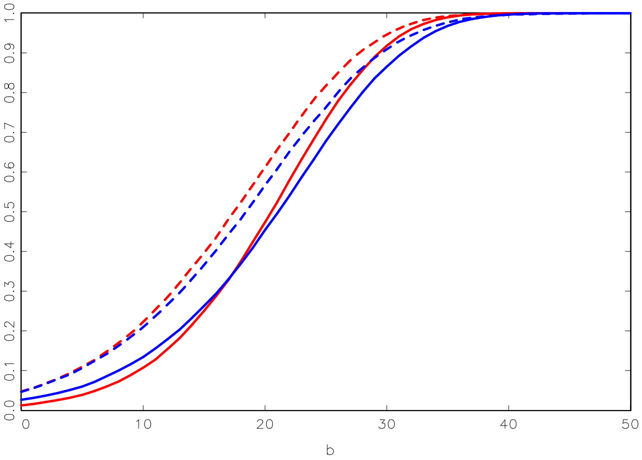
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -40$, $\delta = -0.95$, $\kappa = 0.5$. Right Tail.

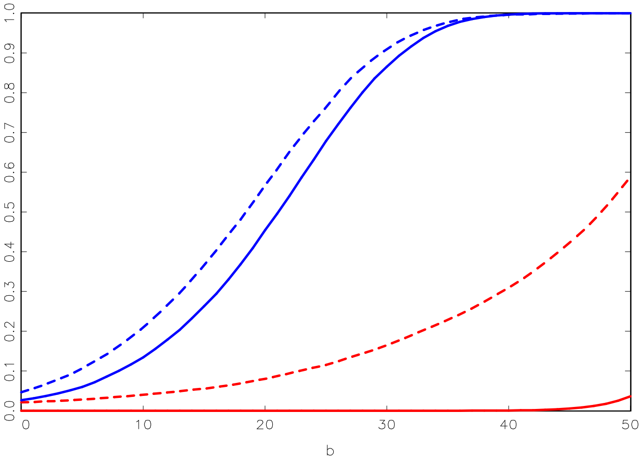


t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——

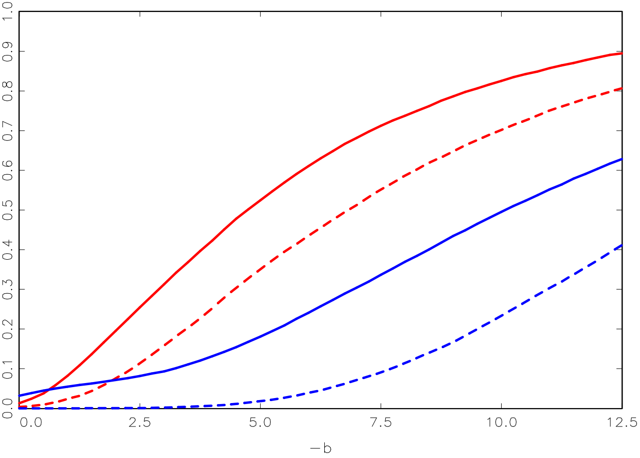
Local Asymptotic Power. $c = -50$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

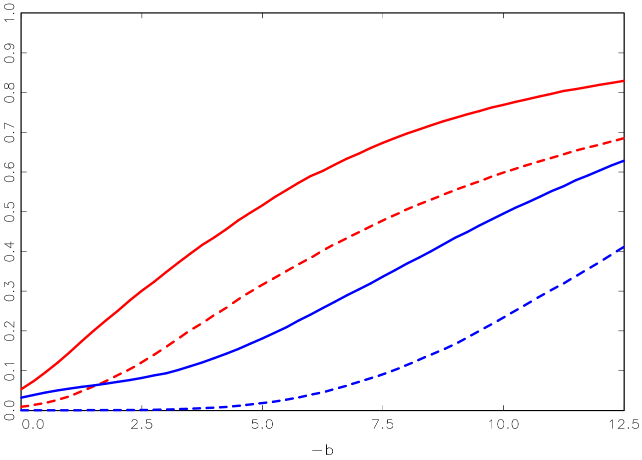


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——



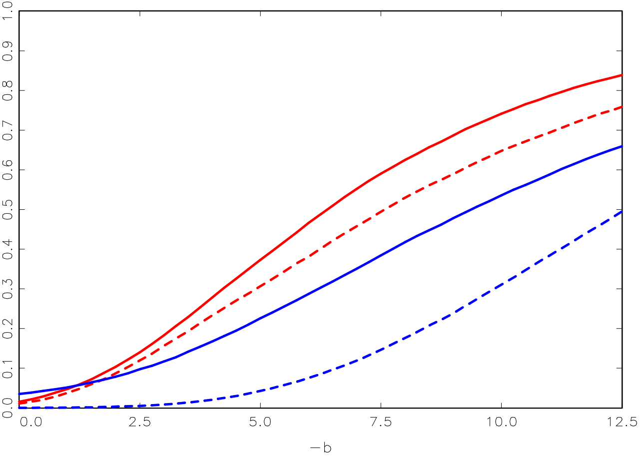
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = 0$, $\delta = -0.95$, $\kappa = 1.0$. Left Tail.



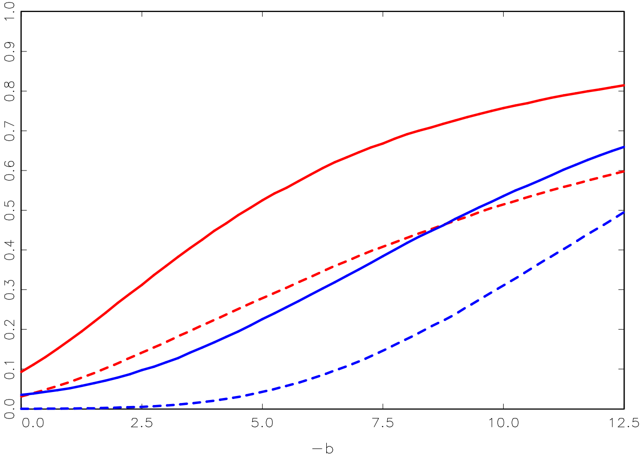
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -2$, $\delta = -0.95$, $\kappa = 0$. Left Tail.

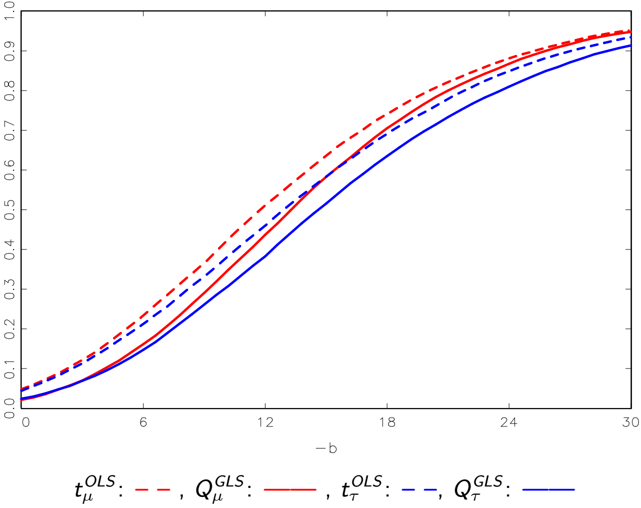


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -2$, $\delta = -0.95$, $\kappa = 1.0$. Left Tail.



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——


0.0
0.1
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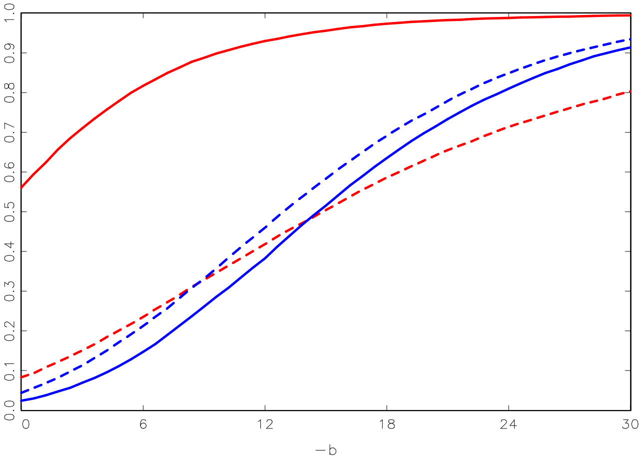
12
18
24
30
−b
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

Local Asymptotic Power. $c = -20$, $\delta = -0.95$, $\kappa = 0.5$. Left Tail.



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——

- We see that for $\kappa = 0$ the mean-only tests almost always outperform their trend-augmented counterparts, as expected.
- For $\kappa \neq 0$ the mean-only tests are unreliable, being asymptotically undersized and correspondingly lacking in power when testing in the right tail, and asymptotically oversized in the left tail.
- For right tailed testing, we ideally would want to use Q_{μ}^{GLS} when $\kappa = 0$ and Q_{τ}^{GLS} when $\kappa \neq 0$ when c is small.
- It can be seen, however, that t_{τ}^{OLS} can outperform Q_{τ}^{GLS} for more negative values of c (the crossing over of the superiority of their asymptotic power functions occurs at roughly $c = 30$) when testing in the right tail. These findings will later motivate our proposed hybrid tests.
- For left tailed testing things are slightly simpler. The asymptotic oversize of the mean-only tests renders them of little use when testing in the left tail, so our hybrid test procedures in this instance will be functions only of trend-augmented tests.

Proposed Right Tailed Tests

Our first proposed testing strategy is a union-of-rejections test, U, defined by the decision rule

$$U: \text{ Reject } H_0 \text{ if } \underline{U} > 0 \tag{4}$$

where

$$\underline{U} := \max\left(\underline{\beta}_{\mu}^{Q}, \underline{\beta}_{\tau}^{Q}\right) \tag{5}$$

where $\underline{\beta}_{\mu}^{Q}$ and $\underline{\beta}_{\tau}^{Q}$ denote the lower bounds of the confidence interval for β obtained from the Q_{μ}^{GLS} and Q_{τ}^{GLS} tests, respectively.

Due to the usual multiple testing problem, when performing this test we scale the significance levels α_1 and α_2 used to construct the confidence intervals for ρ and β in the underlying Q_{μ}^{GLS} and Q_{τ}^{GLS} tests by a constant $\xi < 1$, with ξ chosen such that, for a given value of δ, the asymptotic size of U is no greater than 5% for $c \in [-50, 5]$.

Proposed Right Tailed Tests

- The union-of-rejections test, U, will be shown to have excellent asymptotic power properties, tracking closely the power of Q_{μ}^{GLS} when $\kappa = 0$ and that of Q_{τ}^{GLS} for large values of κ.
- As we have seen, for larger negative values of c ($c \leq -30$) we see that the power of t_{τ}^{OLS} tends to be higher than that of Q_{τ}^{GLS}.
- As such, we consider an extra layer to our test procedure where for right-tailed tests the union-of-rejections test is employed when c is estimated to be "small", and the t_{τ}^{OLS} test is employed when c is estimated to be "large".
- To do so, we propose using a data-based estimate of c to choose which test to perform. Specifically, we propose computing an estimate, $\hat{c}$, that is equal to the with-trend ADF-GLS normalised bias unit root test statistic, henceforth denoted $NB\text{-}GLS_{\tau}$.

It can be shown that the limiting distribution of $\hat{c}$ is given by

$$\hat{c} \xrightarrow{w} \frac{W_{1c}^{\tau,\overline{c}}(1)^2 - 1}{2\int_0^1 W_{1c}^{\tau,\overline{c}}(r)^2 dr}. \tag{6}$$

While it is clear that $\hat{c}$ is not a consistent estimate of c, a near monotonic relationship nonetheless exists between the expected value of the limiting distribution of $\hat{c}$ and the true value of c. We therefore propose a cut-off rule where we employ the U test for $\hat{c} \geq c_R$, but switch to the t_τ^{OLS} test for $\hat{c} < c_R$ for some cut-off point c_R (R denoting right-tailed). Formally, our second proposed testing procedure, S, is therefore given by:

$$S: \text{ Reject } H_0 \text{ if } \underline{US} > 0 \tag{7}$$

where

$$\underline{US} := \mathbb{I}(\hat{c} \geq c_R)\underline{U} + \mathbb{I}(\hat{c} < c_R)\underline{\beta}_\tau^t. \tag{8}$$

and where $\underline{\beta}_\tau^t$ denotes the lower bound of the CI for β from the t_τ^{OLS} test and $\mathbb{I}(.)$ denotes the indicator function equal to 1(0) when its argument is true (false).

Proposed Right Tailed Tests

- Our choice of the cut-off value c_R to use in practice is motivated by the asymptotic local power of the tests, where we found that the local asymptotic power of the U test is superior to that of t_τ for $c \geq -30$, whereas for $c < -30$ the reverse is true.
- We found through extensive Monte Carlo simulation that the choice of $c_R = -35$ gave an overall test for predictability with the best overall power properties, tracking the power of U for small c and that of t_τ^{OLS} for large c.
- We also found that using the existing calibration for U and t_τ^{OLS} led to S maintaining a maximum asymptotic size of 0.05 for $c \in [-50, 5]$, so that no further calibration was required for this particular test.

Proposed Left Tailed Tests

- We propose a simpler strategy for left-tailed tests as the asymptotic oversize of Q_μ^{GLS} and t_μ^{OLS} when $\kappa \neq 0$ prevents the implementation of an asymptotically size-controlled union-of-rejections procedure.
- Examining the relative power of Q_τ^{GLS} and t_τ^{OLS} we found that the Q_τ^{GLS} test only offers superior power to t_τ^{OLS} when c is small, with the power of t_τ^{OLS} above that of Q_τ^{GLS} for even modest values of c.
- As such, for the switching strategy S we propose a simpler version to that used for right-tailed testing where the Q_τ^{GLS} test is employed when $\hat{c} \geq c_L$ (L denoting left-tailed) and the t_τ^{OLS} test is used when $\hat{c} < c_L$.

Proposed Left Tailed Tests

Specifically, for left tailed tests the decision rule for our test procedure S is given by.

$$S: \text{ Reject } H_0 \text{ if } \overline{S} < 0 \tag{9}$$

where

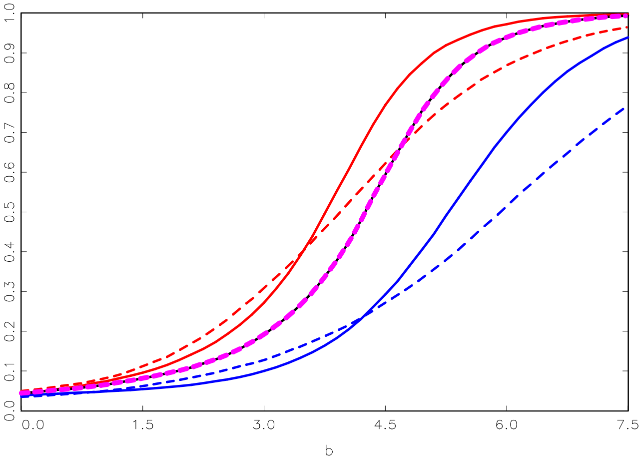
$$\overline{S} := \mathbb{I}(\hat{c} \geq c_L)\overline{\beta}_{\tau}^{Q} + \mathbb{I}(\hat{c} < c_L)\overline{\beta}_{\tau}^{t}. \tag{10}$$

and where $\overline{\beta}_{\tau}^{Q}$ and $\overline{\beta}_{\tau}^{t}$ denote the upper bounds of the confidence interval for β obtained from the Q_{τ}^{GLS} and t_{τ}^{OLS} tests, respectively.
Through extensive Monte Carlo simulations we found that a value of $c_L = -15$ led to a test with the best overall power properties. As was the case for right-tailed testing, we found that the maximum asymptotic size of S was still maximised at 0.05 for $c \in [-50, 5]$ when testing in the left tail, so that again no further calibration was required.

- The U and S tests are constructed under the assumption that the predictor is strongly persistent. When Assumption W holds, such that the predictor is weakly persistent, the Q_τ^{GLS} and t_τ^{OLS} tests, and hence the U and S tests, are asymptotically invalid.
- In contrast, under Assumption W a "conventional" OLS t-test, which compares the OLS t-statistic with standard normal critical values, is asymptotically valid and is optimal (among feasible tests) under Gaussianity, regardless of the value of δ; see Jansson and Moreira (2006,p.704)
- We therefore propose a simple modification to the U and S tests whereby they switch into the standard t-test if there is sufficient evidence that the predictor is weakly persistent.
- To ensure that the standard t-test is used asymptotically when the predictor is weakly persistent we apply the standard t-test instead of the U and S test whenever the with-trend normalised bias ADF statistic applied to the predictor is less than $-\upsilon T^{1/2}$.
- We denote these final testing strategies as U^{hyb} and S^{hyb}

Table of Contents

Local Asymptotic Power. $c = 0$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

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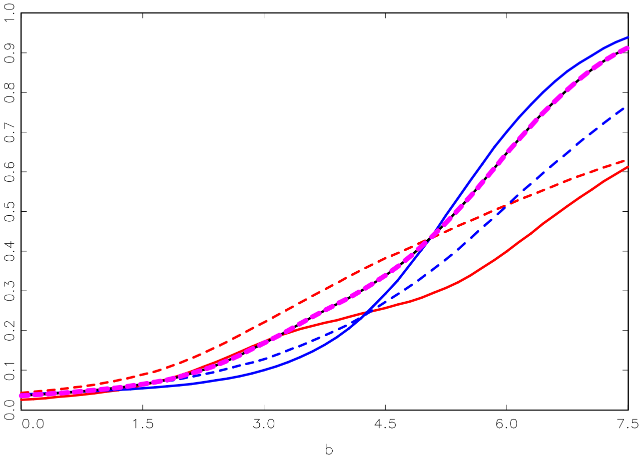

t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

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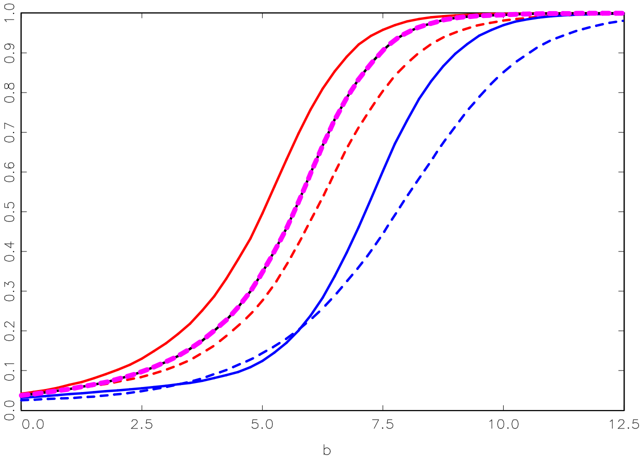

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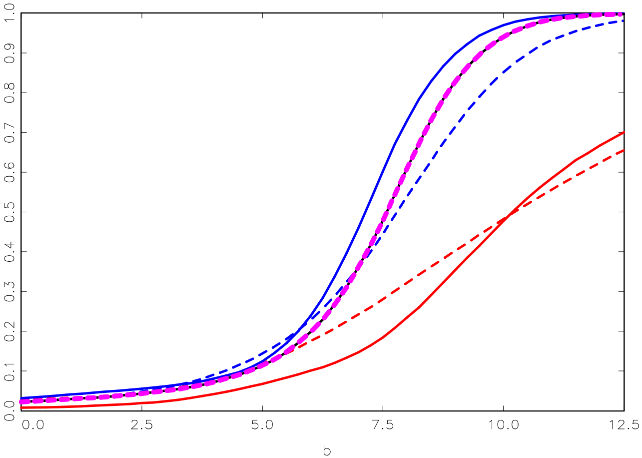
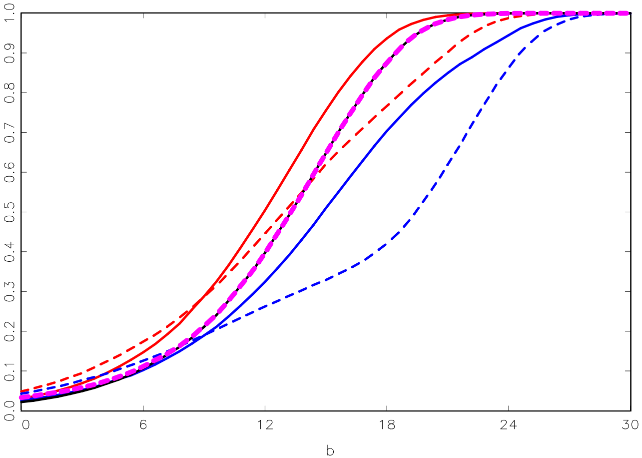


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Local Asymptotic Power. $c = -20$, $\delta = -0.95$, $\kappa = 0$. Right Tail.



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

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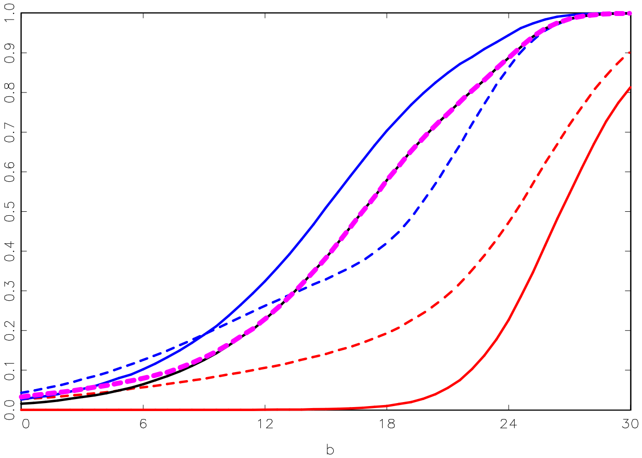

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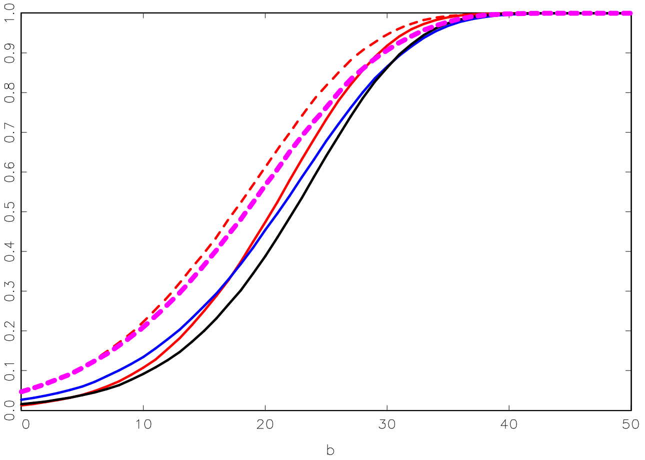

t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

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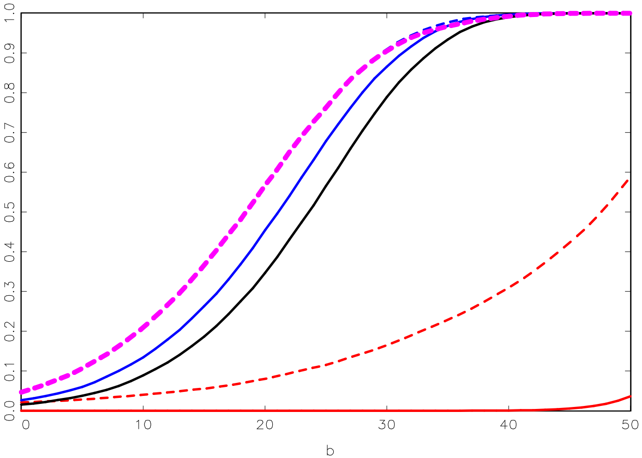

t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

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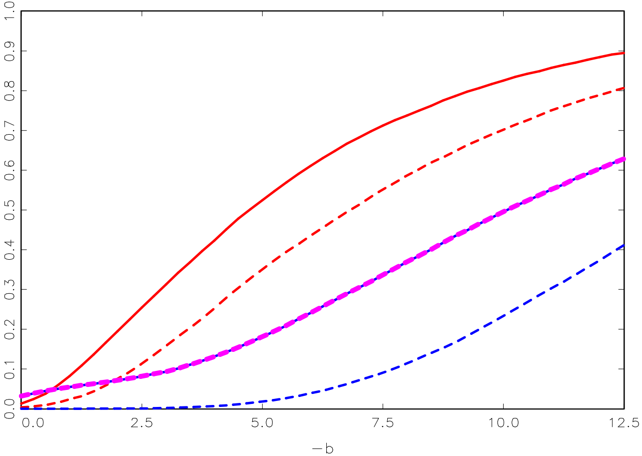

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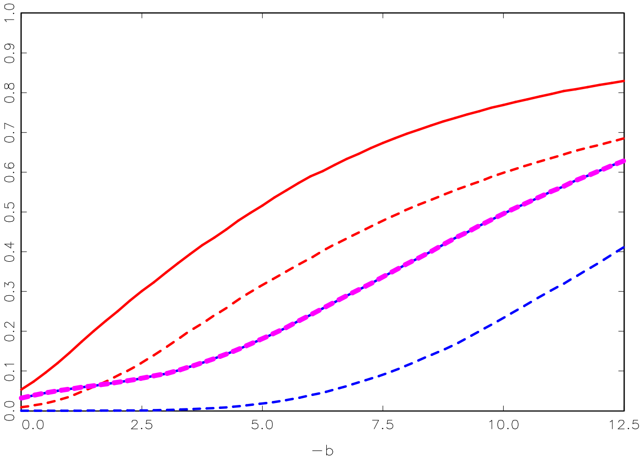

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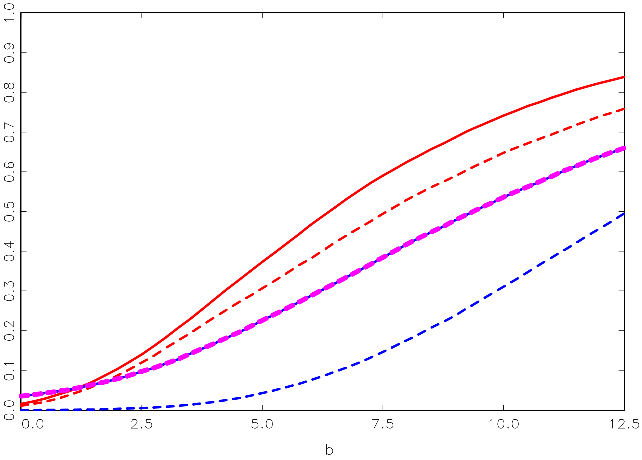

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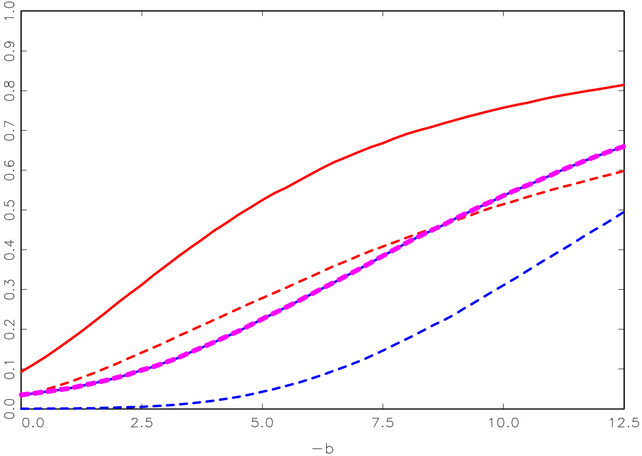
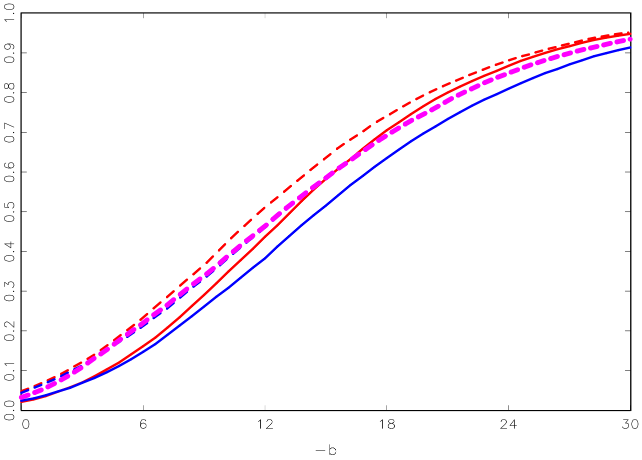


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, S^{hyb}: – –

Local Asymptotic Power. $c = -20$, $\delta = -0.95$, $\kappa = 0$. Left Tail.



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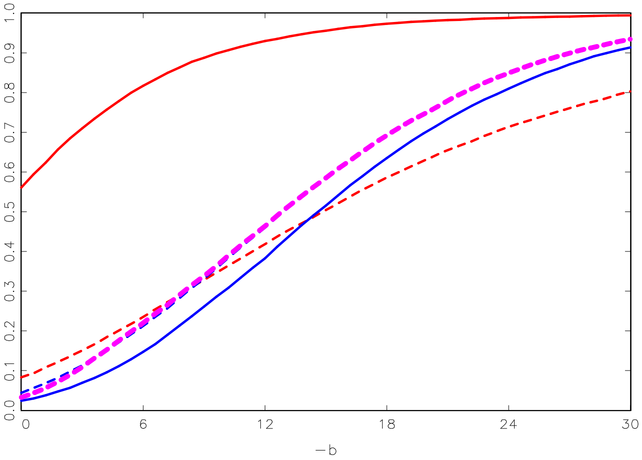

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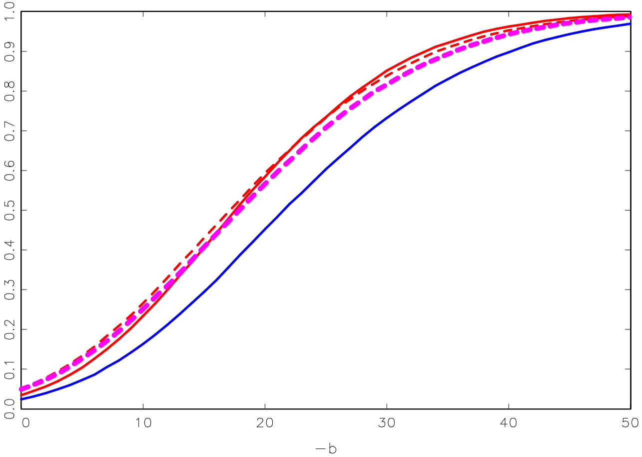

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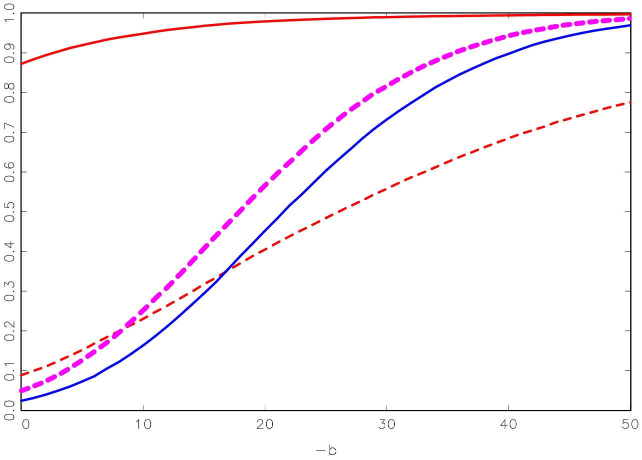


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, S^{hyb}: – –

- Data were generated according to (1) - (3) with $v_t = \phi v_{t-1} + e_t$ where $e_t \sim NIID(0,1)$, setting $w_1 = v_1 = e_1$.
- We set $T = 250$ and generate data according to Assumptions S and T such that $\rho = 1 + cT^{-1}$ and $\gamma_T = \kappa\omega_v T^{-1/2}$, noting that for larger negative values of c the predictor will behave more like a weakly stationary process in finite samples.
- All tests are performed at a nominal level of 0.05. Following CY, lag selection for all of the unit root tests utilised in the test procedures is performed using the Bayes Information Criterion (BIC) with a maximum number of lagged differences of 4.
- Finally we set $\upsilon = 10$ such that our hybrid S^{hyb} and U^{hyb} tests switch into the conventional t-test whenever $NB\text{-}OLS_\tau < -10T^{1/2}$ as we found this choice of υ delivered good finite sample performance across a wide range of DGPs.

Finite Sample Size - $T = 250$, $\phi = 0.0$, $\kappa = 0.0$

		(a) Right Tailed Tests						(b) Left Tailed Tests					
c	δ	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.049	0.048	0.047	0.053	0.055	0.055	0.017	0.005	0.009	0.000	0.009	0.009
	-0.75	0.046	0.049	0.055	0.050	0.056	0.056	0.013	0.010	0.013	0.000	0.013	0.013
0	-0.95	0.049	0.051	0.047	0.036	0.052	0.052	0.010	0.004	0.023	0.000	0.023	0.023
	-0.75	0.052	0.052	0.045	0.038	0.054	0.054	0.010	0.010	0.013	0.000	0.013	0.013
-2	-0.95	0.050	0.044	0.039	0.025	0.044	0.044	0.010	0.011	0.024	0.000	0.024	0.024
	-0.75	0.051	0.038	0.033	0.024	0.044	0.044	0.009	0.018	0.019	0.002	0.019	0.019
-5	-0.95	0.049	0.044	0.036	0.028	0.041	0.042	0.013	0.034	0.016	0.001	0.016	0.016
	-0.75	0.049	0.034	0.031	0.018	0.038	0.038	0.010	0.035	0.013	0.007	0.013	0.013
-10	-0.95	0.045	0.047	0.039	0.039	0.038	0.038	0.019	0.046	0.017	0.006	0.017	0.017
	-0.75	0.044	0.039	0.033	0.020	0.035	0.036	0.012	0.045	0.013	0.026	0.013	0.013
-20	-0.95	0.038	0.049	0.043	0.047	0.034	0.038	0.035	0.046	0.017	0.046	0.017	0.026
	-0.75	0.036	0.045	0.036	0.034	0.031	0.034	0.020	0.046	0.014	0.050	0.014	0.033
-30	-0.95	0.034	0.050	0.053	0.049	0.037	0.047	0.067	0.048	0.019	0.048	0.019	0.048
	-0.75	0.032	0.045	0.041	0.043	0.031	0.040	0.035	0.049	0.014	0.049	0.014	0.049
-40	-0.95	0.032	0.049	0.067	0.050	0.046	0.050	0.107	0.049	0.020	0.047	0.020	0.047
	-0.75	0.030	0.047	0.050	0.046	0.036	0.045	0.057	0.048	0.014	0.048	0.014	0.048
-50	-0.95	0.032	0.049	0.085	0.051	0.060	0.051	0.150	0.048	0.022	0.047	0.022	0.047
	-0.75	0.030	0.048	0.064	0.047	0.046	0.047	0.084	0.048	0.016	0.049	0.016	0.049
-100	-0.95	0.050	0.047	0.316	0.048	0.259	0.048	0.315	0.050	0.040	0.052	0.037	0.052
	-0.75	0.042	0.047	0.248	0.049	0.203	0.049	0.232	0.051	0.026	0.052	0.024	0.052
-250	-0.95	0.281	0.040	0.857	0.036	0.065	0.065	0.447	0.065	0.048	0.074	0.039	0.039
	-0.75	0.274	0.041	0.837	0.038	0.061	0.061	0.407	0.062	0.039	0.067	0.042	0.041

Finite Sample Size - $T = 250$, $\phi = 0.0$, $\kappa = 0.5 + 0.5I(c > -20)$

		(a) Right Tailed Tests						(b) Left Tailed Tests					
c	δ	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.050	0.048	0.047	0.053	0.055	0.055	0.046	0.006	0.009	0.000	0.009	0.009
	-0.75	0.047	0.047	0.055	0.050	0.057	0.057	0.030	0.011	0.013	0.000	0.013	0.013
0	-0.95	0.030	0.044	0.047	0.036	0.044	0.044	0.042	0.011	0.023	0.000	0.023	0.023
	-0.75	0.034	0.045	0.045	0.038	0.043	0.043	0.027	0.017	0.013	0.000	0.013	0.013
-2	-0.95	0.010	0.029	0.039	0.025	0.027	0.027	0.069	0.032	0.024	0.000	0.024	0.024
	-0.75	0.014	0.028	0.033	0.024	0.027	0.027	0.038	0.036	0.019	0.002	0.019	0.019
-5	-0.95	0.001	0.020	0.036	0.028	0.023	0.023	0.161	0.064	0.016	0.001	0.016	0.016
	-0.75	0.004	0.020	0.031	0.018	0.021	0.021	0.077	0.061	0.013	0.007	0.013	0.013
-10	-0.95	0.000	0.011	0.039	0.039	0.024	0.025	0.302	0.101	0.017	0.006	0.017	0.017
	-0.75	0.002	0.012	0.033	0.020	0.022	0.023	0.138	0.085	0.013	0.026	0.013	0.013
-20	-0.95	0.001	0.029	0.043	0.047	0.027	0.037	0.341	0.079	0.017	0.046	0.017	0.026
	-0.75	0.003	0.027	0.036	0.034	0.023	0.029	0.162	0.069	0.014	0.050	0.014	0.033
-30	-0.95	0.001	0.028	0.053	0.049	0.035	0.047	0.460	0.080	0.019	0.048	0.019	0.048
	-0.75	0.002	0.026	0.041	0.043	0.028	0.040	0.229	0.071	0.014	0.049	0.014	0.049
-40	-0.95	0.001	0.026	0.067	0.050	0.045	0.050	0.535	0.081	0.020	0.047	0.020	0.047
	-0.75	0.002	0.026	0.050	0.046	0.035	0.045	0.282	0.072	0.014	0.048	0.014	0.048
-50	-0.95	0.000	0.024	0.085	0.051	0.060	0.051	0.577	0.082	0.022	0.047	0.022	0.047
	-0.75	0.001	0.024	0.064	0.047	0.046	0.047	0.321	0.072	0.016	0.049	0.016	0.049
-100	-0.95	0.001	0.007	0.316	0.048	0.259	0.048	0.609	0.098	0.040	0.052	0.037	0.052
	-0.75	0.002	0.011	0.248	0.049	0.203	0.049	0.377	0.086	0.026	0.052	0.024	0.052
-250	-0.95	0.000	0.003	0.857	0.036	0.065	0.065	0.830	0.158	0.048	0.074	0.039	0.039
	-0.75	0.000	0.006	0.837	0.038	0.061	0.061	0.594	0.125	0.039	0.067	0.042	0.041

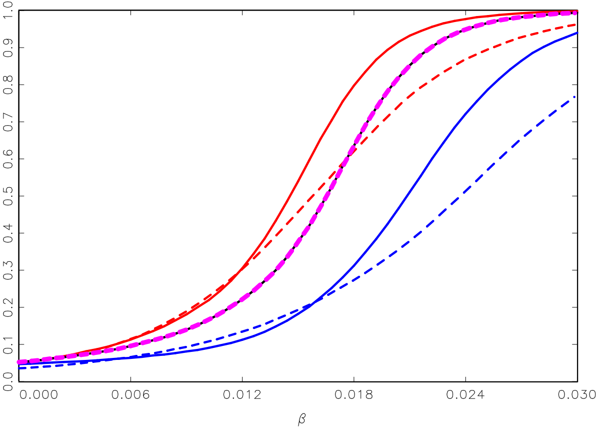
Finite Sample Size - $T = 250$, $\phi = 0.5$, $\kappa = 0.0$.

		(a) Right Tailed Tests						(b) Left Tailed Tests					
c	δ	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}	Q_μ^{GLS}	t_μ^{OLS}	Q_τ^{GLS}	t_τ^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.050	0.050	0.048	0.055	0.054	0.054	0.000	0.006	0.010	0.000	0.010	0.010
	-0.75	0.045	0.051	0.053	0.053	0.056	0.056	0.005	0.010	0.013	0.000	0.013	0.013
0	-0.95	0.048	0.051	0.046	0.038	0.051	0.052	0.010	0.004	0.021	0.000	0.021	0.021
	-0.75	0.051	0.051	0.044	0.040	0.054	0.054	0.010	0.010	0.011	0.001	0.011	0.011
-2	-0.95	0.048	0.043	0.037	0.026	0.042	0.043	0.010	0.011	0.023	0.000	0.023	0.023
	-0.75	0.050	0.040	0.033	0.025	0.043	0.043	0.009	0.018	0.018	0.002	0.018	0.018
-5	-0.95	0.046	0.045	0.033	0.028	0.037	0.037	0.013	0.033	0.015	0.001	0.015	0.015
	-0.75	0.046	0.036	0.029	0.018	0.036	0.036	0.010	0.034	0.013	0.006	0.013	0.013
-10	-0.95	0.040	0.048	0.033	0.035	0.031	0.033	0.017	0.046	0.017	0.005	0.017	0.017
	-0.75	0.040	0.039	0.029	0.019	0.030	0.031	0.012	0.043	0.013	0.023	0.013	0.014
-20	-0.95	0.028	0.049	0.032	0.046	0.027	0.037	0.033	0.046	0.020	0.036	0.020	0.026
	-0.75	0.029	0.045	0.028	0.029	0.026	0.030	0.020	0.047	0.015	0.047	0.015	0.030
-30	-0.95	0.021	0.048	0.034	0.047	0.024	0.045	0.063	0.047	0.021	0.049	0.021	0.049
	-0.75	0.023	0.047	0.028	0.037	0.022	0.036	0.034	0.047	0.016	0.049	0.016	0.047
-40	-0.95	0.016	0.049	0.035	0.049	0.024	0.048	0.104	0.048	0.023	0.049	0.023	0.049
	-0.75	0.019	0.047	0.029	0.042	0.022	0.042	0.055	0.048	0.016	0.047	0.016	0.047
-50	-0.95	0.013	0.050	0.037	0.049	0.024	0.049	0.150	0.049	0.024	0.049	0.024	0.049
	-0.75	0.015	0.047	0.030	0.045	0.021	0.045	0.081	0.048	0.017	0.048	0.017	0.048

Finite Sample Size - $T = 250$, $\phi = 0.5$ $\kappa = 0.5 + 0.5I(c > -20)$.

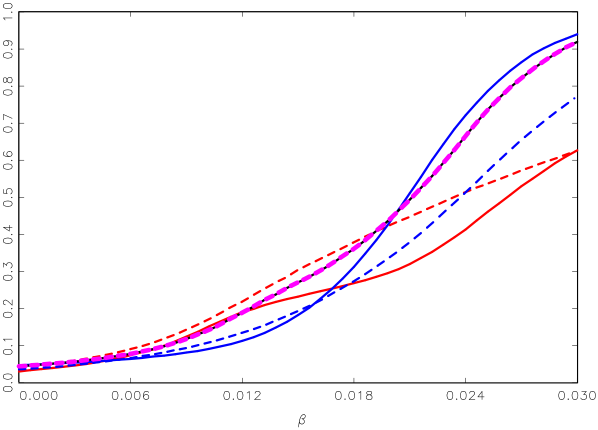
		(a) Right Tailed Tests						(b) Left Tailed Tests					
c	δ	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}	Q_{μ}^{GLS}	t_{μ}^{OLS}	Q_{τ}^{GLS}	t_{τ}^{OLS}	U^{hyb}	S^{hyb}
2	-0.95	0.049	0.048	0.048	0.055	0.056	0.056	0.007	0.006	0.010	0.000	0.010	0.010
	-0.75	0.046	0.047	0.053	0.053	0.057	0.057	0.012	0.012	0.013	0.000	0.013	0.013
0	-0.95	0.030	0.044	0.046	0.038	0.044	0.044	0.039	0.011	0.021	0.000	0.021	0.021
	-0.75	0.033	0.043	0.044	0.040	0.044	0.044	0.026	0.016	0.011	0.001	0.011	0.011
-2	-0.95	0.009	0.030	0.037	0.026	0.026	0.026	0.067	0.033	0.023	0.000	0.023	0.023
	-0.75	0.012	0.028	0.033	0.025	0.026	0.027	0.039	0.036	0.018	0.002	0.018	0.018
-5	-0.95	0.001	0.020	0.033	0.028	0.020	0.021	0.155	0.064	0.015	0.001	0.015	0.015
	-0.75	0.004	0.019	0.029	0.018	0.020	0.021	0.077	0.059	0.013	0.006	0.013	0.013
-10	-0.95	0.001	0.011	0.033	0.035	0.020	0.022	0.286	0.098	0.017	0.005	0.017	0.017
	-0.75	0.003	0.012	0.029	0.019	0.019	0.021	0.134	0.084	0.013	0.023	0.013	0.014
-20	-0.95	0.000	0.025	0.032	0.046	0.021	0.035	0.339	0.084	0.020	0.036	0.020	0.026
	-0.75	0.002	0.022	0.028	0.029	0.019	0.026	0.161	0.073	0.015	0.047	0.015	0.030
-30	-0.95	0.000	0.020	0.034	0.047	0.021	0.045	0.452	0.090	0.021	0.049	0.021	0.049
	-0.75	0.002	0.018	0.028	0.037	0.019	0.035	0.222	0.078	0.016	0.049	0.016	0.047
-40	-0.95	0.000	0.015	0.035	0.049	0.023	0.048	0.514	0.095	0.023	0.049	0.023	0.049
	-0.75	0.002	0.014	0.029	0.042	0.020	0.042	0.262	0.082	0.016	0.047	0.016	0.047
-50	-0.95	0.001	0.011	0.037	0.049	0.024	0.049	0.542	0.099	0.024	0.049	0.024	0.049
	-0.75	0.002	0.012	0.030	0.045	0.020	0.045	0.286	0.084	0.017	0.048	0.017	0.048

Finite Sample Power - Right Tail. $c = 0$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



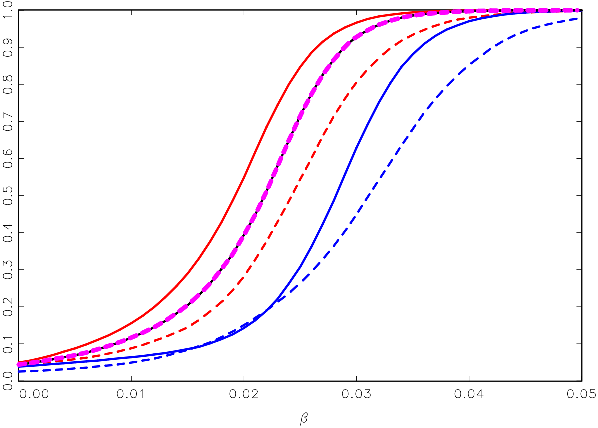
t_μ^{OLS}: – –, Q_μ^{GLS}: ——, t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = 0$, $\kappa = 1.0$, $\delta = -0.95$, $\phi = 0.0$



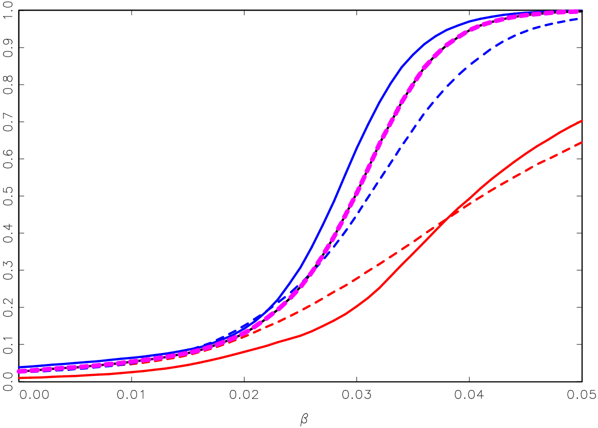
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -2$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



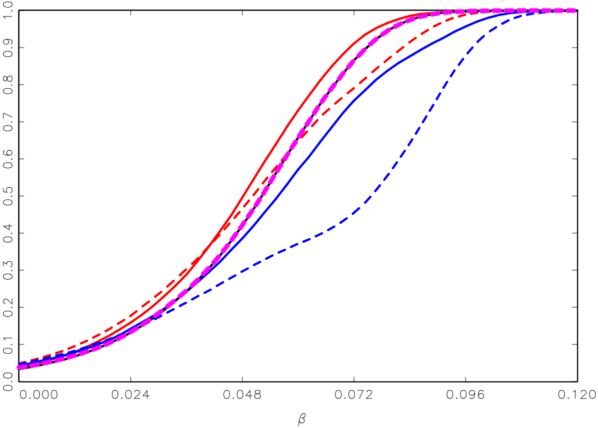
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -2$, $\kappa = 1.0$, $\delta = -0.95$, $\phi = 0.0$



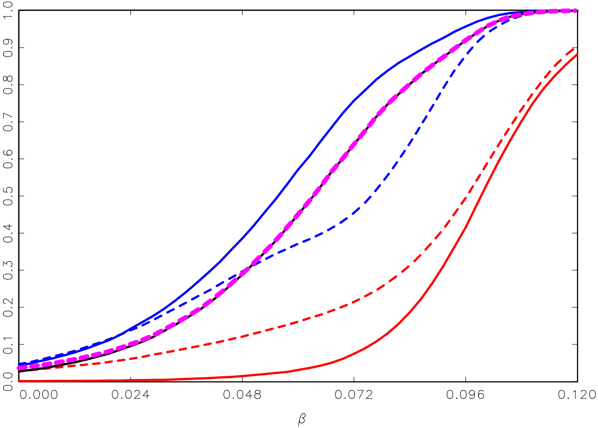
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -20$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



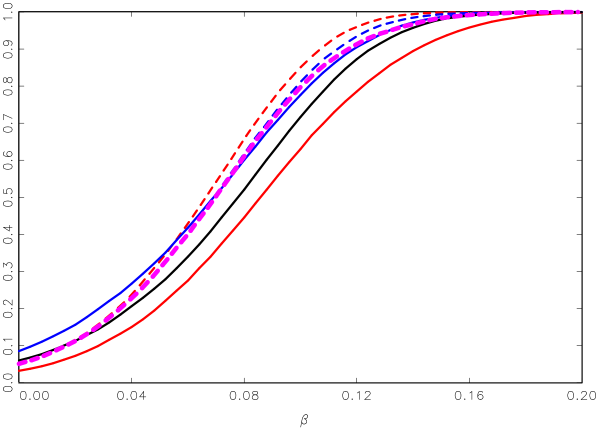
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -20$, $\kappa = 0.5$, $\delta = -0.95$, $\phi = 0.0$



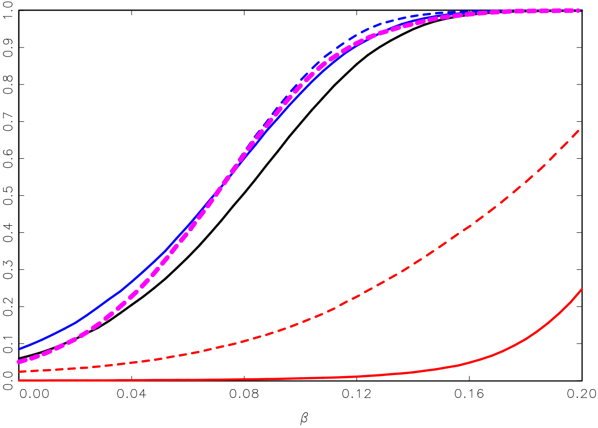
t_{μ}^{OLS}: – –, Q_{μ}^{GLS}: ——, t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -50$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



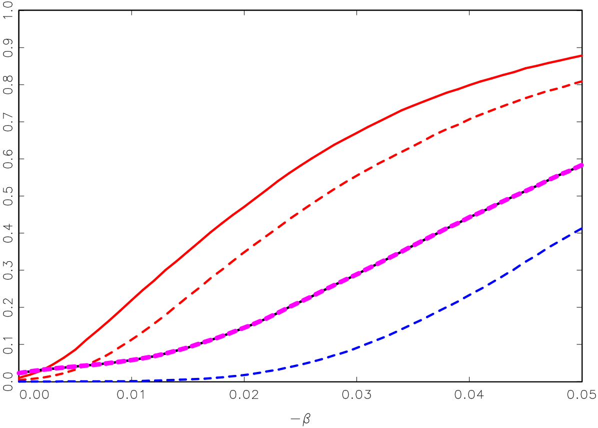
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Right Tail. $c = -50$, $\kappa = 0.5$, $\delta = -0.95$, $\phi = 0.0$



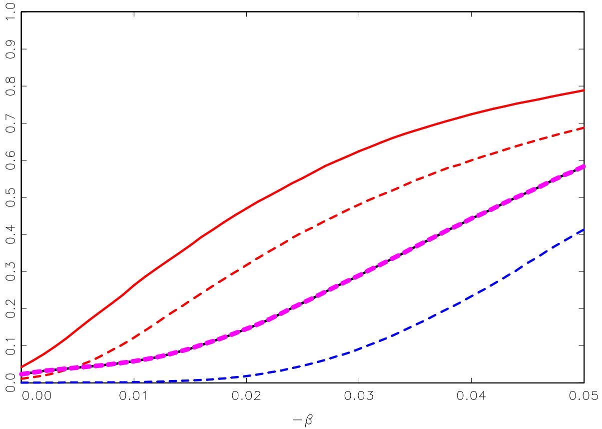
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = 0$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



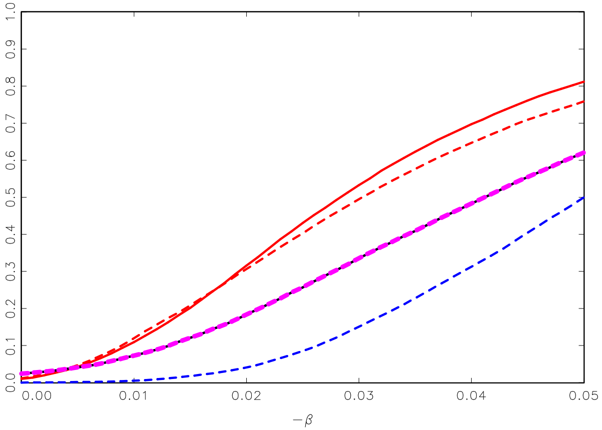
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = 0$, $\kappa = 1.0$, $\delta = -0.95$, $\phi = 0.0$

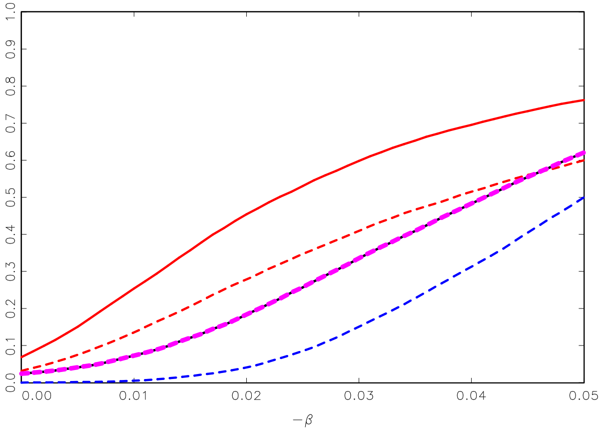


t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = -2$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$

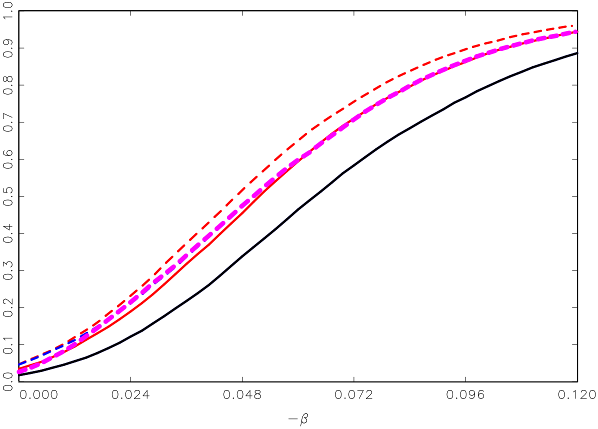


t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –



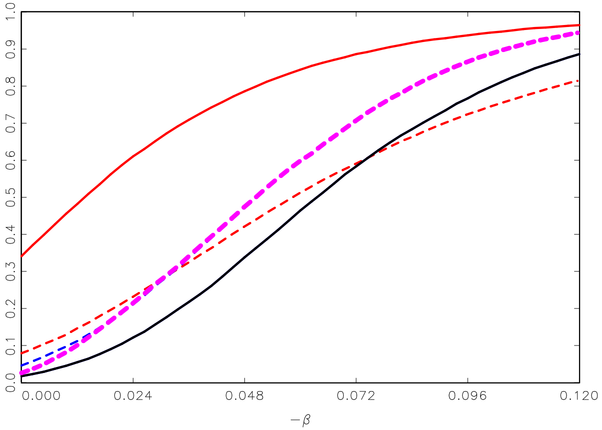
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = -20$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



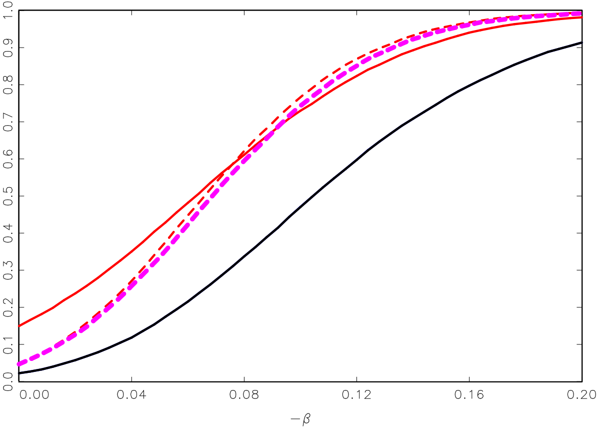
t_μ^{OLS}: – –, Q_μ^{GLS}: ——, t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = -20$, $\kappa = 0.5$, $\delta = -0.95$, $\phi = 0.0$



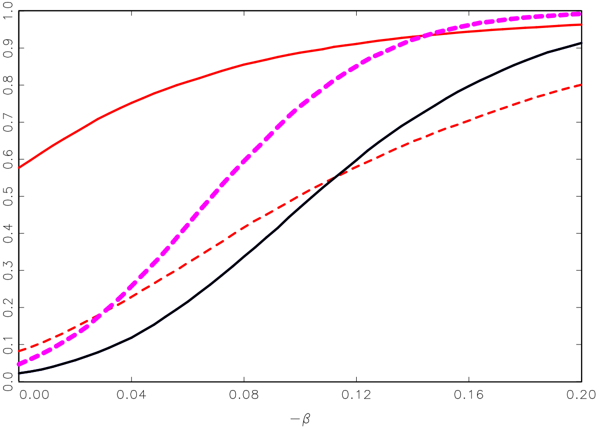
t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = -50$, $\kappa = 0.0$, $\delta = -0.95$, $\phi = 0.0$



t_{μ}^{OLS}: – – , Q_{μ}^{GLS}: —— , t_{τ}^{OLS}: – –, Q_{τ}^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Finite Sample Power - Left Tail. $c = -50$, $\kappa = 0.5$, $\delta = -0.95$, $\phi = 0.0$



t_μ^{OLS}: – – , Q_μ^{GLS}: —— , t_τ^{OLS}: – –, Q_τ^{GLS}: ——, U^{hyb}: ——, S^{hyb}: – –

Table of Contents

- We apply the tests for predictability outlined in this paper to the US equity series analysed in Welch and Goyal (2008), using updated data at all available data frequencies (annual, quarterly and monthly) for the period 1926-2021.
- The dependent variable, r_t, is the S&P500 value-weighted log-return, and for x_t we consider the same thirteen candidate predictors variables as Harvey *et al.* (2021): the dividend payout ratio, earnings-price ratio, dividend-price ratio, dividend yield, default yield spread, long-term yield, default return spread, net equity expansion, inflation rate, Treasury bill rate, term spread the book-to-market ratio and stock variance.

Trend Tests

- We first formally test for the presence of a linear trend in each predictor using a range of trend tests available in the literature that are designed to be robust to whether Assumption S or W holds; namely the $t_{\beta}^{RQF}(MU)$ test of Perron and Yabu (2009), the z_{λ}, z_{λ}^{m1} and z_{λ}^{m2} tests of Harvey *et al.* (2007), and the *Dan-J* test of Bunzel and Vogelsang (2005).
- We perform left-tailed trend tests for all predictors with the exception of the inflation rate and term spread for which right-tailed tests are performed, using the setting recommended by the authors in each case.
- For each of the default yield spread, long term yield, default return spread, inflation rate, treasury bill rate and stock variance no trend is detected, regardless of data frequency. In contrast, for the dividend payout ratio, earnings-price ratio, dividend-price ratio, dividend yield and net equity expansion series a significant linear trend is detected regardless of the data frequency. For the remaining predictors the results of the trend tests are mixed, with the trend tests indicating the presence of a trend at some, but not all, data frequencies. In summary, there is at least some statistically significant evidence of a linear trend being present in the majority of the predictors considered.

- The following Tables report the lower bound of the confidence interval for β, denoted generically as $\underline{\beta}$, for each predictor at each frequency, and for each of the predictability tests discussed in the paper.
- Also reported is the estimator $\hat{\delta}$ from the with-trend Bonferroni type test procedures. We highlight any instances where this lower bound is greater than zero in bold to help identify instances where the null of $\beta = 0$ is rejected in favour of the alternative that $\beta > 0$.
- Finally, for the lower bound of β from the S^{hyb} and U^{hyb} tests we use the superscript z to identify instances where these tests have switched into the conventional t-test, and for S^{hyb} we use the superscript t to denote instances where this test is basing inference on the t_{τ}^{OLS} test.

Tests For Predictability - Annual Data

							$\underline{\beta}$					
Predictor	$p(t_{\beta}^{RQF})$	$p(Z_{\lambda})$	Z_{λ}^{m1}	Z_{λ}^{m2}	*DAN-J*	$\hat{\delta}$	t_{μ}^{OLS}	Q_{μ}^{GLS}	t_{τ}^{OLS}	Q_{τ}^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	**0.000**	**0.000**	$***$	$***$	$***$	-0.313	-0.1650	-0.1439	-0.1635	-0.1353	-0.1605	-0.1635^{t}
Earnings-Price Ratio	**0.083**	0.223				-0.301	-0.0018	-0.0095	**0.0192**	**0.0391**	**0.0188**	**0.0188**
Dividend-Price Ratio	**0.000**	**0.008**	$**$	$**$	$*$	-0.843	-0.0624	-0.0443	-0.0280	**0.0406**	**0.0251**	**0.0251**
Dividend Yield	0.200	**0.028**	$**$	$**$	$*$	0.133	-0.0024	-0.0024	**0.0838**	**0.0753**	**0.0753**	**0.0753**
Default Yield Spread	0.314	0.397				-0.570	-0.1221	-0.1002	-0.1705	-0.1242	-0.1192	-0.1192
Long Term Yield	0.376	0.384				-0.028	-0.0467	-0.0460	-0.0547	-0.0539	-0.0544	-0.0544
Default Return Spread	0.267	0.227				0.315	-0.2064	-0.2143	-0.1993	-0.4130	-1.0099^{z}	-1.0099^{z}
Net Equity Expansion	**0.000**	**0.000**	$***$	$***$	$**$	0.086	-0.3479	-0.3280	-0.3408	-0.3512	-0.3512	-0.3408^{t}
Inflation Rate	0.259	0.262				-0.032	-0.0724	-0.0738	-0.0834	-0.0872	-0.0943	-0.0943
Treasury Bill Rate	0.355	0.363				0.093	-0.0802	-0.0859	-0.0817	-0.0886	-0.0886	-0.0886
Term Spread	0.361	0.363				-0.111	-0.0821	-0.0837	-0.0943	-0.0905	-0.1065	-0.0943^{t}
Book-to-market Ratio	0.196	0.282				-0.806	-0.0388	**0.0036**	-0.0313	-0.0229	-0.0087	-0.0087
Stock Variance	0.306	0.472				0.398	-0.1521	-0.2169	-0.1559	-0.2288	-0.2288	-0.1559^{t}

Notes:

(i) The entries in the columns headed $p(t_{\beta}^{RQF})$ and $p(Z_{\lambda})$ denote *p*-values for the t_{β}^{RQF} and Z_{λ} tests. Bold entries highlight *p*-values below 0.1.

(ii) For Z_{λ}^{m1}, Z_{λ}^{m2} and *DAN-J*, $*$ denotes rejection at the 10% level, $**$ denotes rejection at the 5% level, and $***$ denotes rejection at the 1% level.

(iii) Bold entries in the $\underline{\beta}$ columns highlight cases where the null hypothesis of no predictability can be rejected at the 5% level.

(iv) For entries in the U^{hyb} and S^{hyb} columns, a z superscript denotes that the test compares t_{τ} with $N(0, 1)$ critical values, while a t superscript denotes that the test bases inference on the t_{τ}^{OLS} test.

(v) In the case of Stock Variance, we report $\hat{\delta}$ and $\underline{\beta}$ for (-1) $\times$ Stock Variance as the predictor. A right-tailed test from this regression is equivalent to a left-tailed test using the original data.

							$\underline{\beta}$					
Predictor	$p(t_{\beta}^{RQF})$	$p(Z_{\lambda})$	Z_{λ}^{m1}	Z_{λ}^{m2}	*DAN-J*	$\hat{\delta}$	t_{μ}^{OLS}	Q_{μ}^{GLS}	t_{τ}^{OLS}	Q_{τ}^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	**0.000**	**0.064**	$**$	$**$	$***$	-0.120	-0.0229	-0.0235	-0.0250	-0.0307	-0.0277	-0.0250^{t}
Earnings-Price Ratio	**0.000**	0.241			$*$	-0.614	**0.0043**	-0.0157	**0.0068**	**0.0049**	**0.0006**	$\mathbf{0.0068}^{t}$
Dividend-Price Ratio	0.183	**0.072**	$**$	$**$	$**$	-0.949	-0.0085	-0.0061	-0.0002	**0.0160**	**0.0110**	**0.0110**
Dividend Yield	0.204	0.106	$*$	$**$	$*$	0.113	**0.0023**	**0.0024**	**0.0240**	**0.0224**	**0.0224**	$\mathbf{0.0240}^{t}$
Default Yield Spread	0.196	0.416				-0.512	**0.0095**	**0.0126**	-0.0051	**0.0062**	**0.0071**	**0.0071**
Long Term Yield	0.363	0.391				-0.054	-0.0159	-0.0155	-0.0171	-0.0169	-0.0177	-0.0177
Default Return Spread	0.355	0.265				0.303	-0.1263	-0.1792	-0.1247	-0.3683	-0.6196^{z}	-0.6196^{z}
Net Equity Expansion	**0.000**	**0.004**	$***$	$***$	$**$	0.115	-0.0755	-0.0676	-0.0909	-0.0898	-0.0898	-0.0909^{t}
Inflation Rate	0.151	0.314				0.030	-0.1078	-0.1020	-0.1093	-0.1033	-0.1033	-0.1093^{t}
Treasury Bill Rate	0.393	0.414				-0.067	-0.0280	-0.0269	-0.0298	-0.0295	-0.0303	-0.0303
Term Spread	**0.074**	0.345				0.031	-0.0273	-0.0256	-0.0279	-0.0279	-0.0279	-0.0279^{t}
Book-to-market Ratio	0.409	0.360			$*$	-0.793	**0.0212**	**0.0185**	**0.0392**	**0.0122**	**0.0153**	**0.0153**
Stock Variance	0.227	0.455				0.290	-0.1263	-0.1100	-0.1272	-0.1320	-0.1320	-0.1272^{t}

Notes:

(i) The entries in the columns headed $p(t_{\beta}^{RQF})$ and $p(Z_{\lambda})$ denote p-values for the t_{β}^{RQF} and Z_{λ} tests. Bold entries highlight p-values below 0.1.

(ii) For Z_{λ}^{m1}, Z_{λ}^{m2} and *DAN-J*, $*$ denotes rejection at the 10% level, $**$ denotes rejection at the 5% level, and $***$ denotes rejection at the 1% level.

(iii) Bold entries in the $\underline{\beta}$ columns highlight cases where the null hypothesis of no predictability can be rejected at the 5% level.

(iv) For entries in the U^{hyb} and S^{hyb} columns, a z superscript denotes that the test compares t_{τ} with $N(0,1)$ critical values, while a t superscript denotes that the test bases inference on the t_{τ}^{OLS} test.

(v) In the case of Stock Variance, we report $\hat{\delta}$ and $\underline{\beta}$ for (-1)$\times$ Stock Variance as the predictor. A right-tailed test from this regression is equivalent to a left-tailed test using the original data.

Predictor	$p(t_{\beta}^{RQF})$	$p(Z_{\lambda})$	Z_{λ}^{m1}	Z_{λ}^{m2}	$DAN\text{-}J$	$\hat{\delta}$	$\underline{\beta}$ t_{μ}^{OLS}	Q_{μ}^{GLS}	t_{τ}^{OLS}	Q_{τ}^{GLS}	U^{hyb}	S^{hyb}
Dividend Payout Ratio	0.412	**0.089**	$**$	$***$	$**$	-0.049	-0.0052	-0.0053	-0.0059	-0.0065	-0.0059	-0.0059^{t}
Earnings-Price Ratio	0.374	0.353		$**$		-0.799	**0.0009**	-0.0060	**0.0015**	**0.0017**	**0.0006**	$\mathbf{0.0015}^{t}$
Dividend-Price Ratio	0.200	0.198	$*$	$**$	$*$	-0.975	-0.0042	-0.0031	-0.0026	-0.0008	-0.0026	-0.0026
Dividend Yield	0.201	0.199	$*$	$**$	$*$	-0.068	**0.0008**	**0.0008**	**0.0096**	**0.0099**	**0.0085**	**0.0085**
Default Yield Spread	0.438	0.434				-0.248	-0.0015	-0.0012	-0.0036	-0.0017	-0.0028	-0.0028
Long Term Yield	0.354	0.384				-0.087	-0.0051	-0.0049	-0.0057	-0.0057	-0.0056	-0.0056
Default Return Spread	0.321	0.374				0.182	-0.0162	**0.0636**	-0.0158	-0.0259	-0.0678^{z}	-0.0678^{z}
Net Equity Expansion	**0.000**	**0.020**	$**$	$***$	$**$	-0.030	-0.0222	-0.0225	-0.0268	-0.0269	-0.0240	-0.0240
Inflation Rate	0.189	0.456				0.035	-0.0747	-0.0652	-0.0760	-0.0720	-0.0720	-0.0760^{t}
Treasury Bill Rate	0.387	0.402				-0.056	-0.0072	-0.0070	-0.0077	-0.0077	-0.0079	-0.0079
Term Spread	**0.001**	0.400		$*$		0.008	-0.0079	-0.0078	-0.0084	-0.0084	-0.0084	-0.0084^{t}
Book-to-market Ratio	0.438	0.435			$*$	-0.807	-0.0001	**0.0035**	**0.0007**	-0.0023	**0.0025**	**0.0025**
Stock Variance	0.204	0.497				0.267	-0.0429	-0.0018	-0.0431	-0.0130	-0.0130	-0.0431^{t}

Notes:

(i) The entries in the columns headed $p(t_{\beta}^{RQF})$ and $p(Z_{\lambda})$ denote p-values for the t_{β}^{RQF} and Z_{λ} tests. Bold entries highlight p-values below 0.1.

(ii) For Z_{λ}^{m1}, Z_{λ}^{m2} and $DAN\text{-}J$, $*$ denotes rejection at the 10% level, $**$ denotes rejection at the 5% level, and $***$ denotes rejection at the 1% level.

(iii) Bold entries in the $\underline{\beta}$ columns highlight cases where the null hypothesis of no predictability can be rejected at the 5% level.

(iv) For entries in the U^{hyb} and S^{hyb} columns, a z superscript denotes that the test compares t_{τ} with $N(0,\, 1)$ critical values, while a t superscript denotes that the test bases inference on the t_{τ}^{OLS} test.

(v) In the case of Stock Variance, we report $\hat{\delta}$ and $\underline{\beta}$ for (-1) $\times$ Stock Variance as the predictor. A right-tailed test from this regression is equivalent to a left-tailed test using the original data.

- For the dividend payout ratio, long term yield, net equity expansion, inflation rate, Treasury bill rate, term spread and stock variance predictors, no evidence of predictability is found by any of the tests for any data frequency and so we will not discuss results for these predictors further.
- For the earnings-price ratio, as noted above, a linear trend was detected at all frequencies, giving reasonable evidence that a trend is present in this predictor. For this predictor the Q_{τ}^{GLS}, t_{τ}^{OLS}, U^{hyb} and S^{hyb} tests all reject the null of no predictability at each data frequency, while the Q_{μ}^{GLS} test fails to reject at any frequency and the t_{μ}^{OLS} test rejects only at the monthly and quarterly frequencies. These results suggest that for this predictor an unmodelled trend in the predictor may be negatively impacting the power of the constant-only tests, with the trend-augmented and hybrid tests retaining power to find significant evidence of predictability.

- A similar story is seen for the dividend-price ratio where again a trend was detected at each data frequency, and where the Q_{μ}^{GLS} and t_{μ}^{OLS} tests provide no evidence of predictability at any frequency. The Q_{τ}^{GLS}, S^{hyb} and U^{hyb} tests, on the other hand, find evidence of predictability at both the annual and quarterly frequencies, although no predictability is detected by any test at the monthly frequency.
- Turning to the dividend yield predictor a significant trend is detected at each data frequency by at least one of the trend tests and all of the predictability tests find significant evidence of predictability at all data frequencies, with the exception of the Q_{μ}^{GLS} and t_{μ}^{OLS} tests at the annual frequency. Interestingly, the annual frequency data provides the strongest evidence for the presence of a trend among the three data frequencies and so it is noteworthy that it is for the annual data that the Q_{μ}^{GLS} and t_{μ}^{OLS} fail to detect predictability, while our hybrid tests deliver rejections.

- There appears to be no evidence of a trend in the default yield spread, and the only data frequency at which predictability is detected for this predictor is for quarterly data. For quarterly data rejections are found by all but the t_τ^{OLS} test, reflective of the fact that our hybrid S^{hyb} and U^{hyb} tests are competitive on power with the best performing individual tests when no (or a very small) trend is present in the predictor.
- For the default return spread no trend is detected at any data frequency and only one rejection, at the monthly frequency, is observed for the Q_μ^{GLS} test. As the S^{hyb} and U^{hyb} tests have switched into the conventional t test for this predictor it is likely that this predictor is weakly persistent, and that the rejection from Q_μ^{GLS} may be reflecting the oversize of this test for weakly persistent predictors.

- Finally, results for the book-to-market ratio are mixed. At the annual frequency no evidence of a trend is found and the only test to reject is Q_{μ}^{GLS} which is perhaps to be expected if no trend is present and the predictive power of this predictor is weak. At the quarterly and monthly frequencies, however, a trend is detected and all of the tests reject the null of no predictability with the exception of t_{μ}^{OLS} and Q_{τ}^{GLS} for monthly data.
- Overall we find that for several predictor series, a trend appears to be present, and at the same time the constant-only Bonferroni Q and t-tests fail to reject the null of no predictability, indicating that the presence of omitted trends may be negatively impacting the power of the constant-only tests. In contrast, our proposed tests find evidence of predictability in many of these cases, highlighting the value of our hybrid procedures in detecting predictability when uncertainty exists regarding the presence of a linear trend in the predictor.

Table of Contents

- We consider trend-augmented versions of the Bonferroni Q test of CY and the Bonferroni t-test of CES.
- In the presence of an omitted trend in the predictor, when $\delta < 0$ ($\delta > 0$) the constant-only Bonferroni Q and t-tests can be severely undersized when testing in the right (left) tail, displaying a subsequent lack of power, and severely oversized when testing in the left (right) tail.
- Trend-augmented variants of the Bonferroni Q and t-tests, while displaying power below their constant-only counterparts when no trend is present, are invariant to a trend in the predictor.

Conclusion

- We propose union-of-rejections type hybrid testing procedures that are able to capture the power of the constant-only Bonferroni Q test when the predictor admits only a deterministic constant, and the power of the trend-augmented Bonferroni Q and t-tests when a trend is present in the predictor, with S^{hyb} being our recommended testing procedure given that it has controlled size, and is always among the most powerful tests, over the full range of parameter settings considered.
- An empirical illustration using an updated version of the dataset of Welch and Goyal (2008) demonstrated that our proposed approach finds evidence of predictability in several instances where a trend appears to be present in the predictor where the constant-only Bonferroni Q and t-tests fail to reject, indicating that the presence of omitted trends may be negatively impacting the power of the constant-only tests in this very widely used dataset.