# A Bootstrap Stationarity Test for Predictive Regression Invalidity

Robert Taylor

University of Essex

Co-authors:
Iliyan Georgiev (University of Bologna)
David Harvey (University of Nottingham)
Stephen Leybourne (University of Nottingham)

EFiC Conference, 7/7/2017

# Introduction

- Predictive regressions play an important role in empirical economics. For example, Granger causality implies that some variable is not considered as a cause for another variable if the former cannot predict the latter.
- In financial economics it is of interest whether current information on variables such as dividend yields or interest spreads contain information about future (excess) stock price returns (e.g. Campbell and Shiller, 1988, JF).

- Another linear rational expectations hypothesis that can be tested by predictive regression methods is the uncovered interest rate parity hypothesis (UIPH), which asserts that the expected change of future exchange rates is equal to the difference between (conformable) domestic and foreign interest rates. Here the predictive regression is of the changes of the exchange rate minus the previous period interest rate differential regressed onto the previous period interest rate differential.
- If the UIPH holds, interest rate differentials should not be able to predict, but the coefficient on interest rate differentials is often found to be significantly negative in practice (e.g. Froot and Thaler, 1990, Jn. Ec. Persp.)

- An important practical problem with performing such predictive regressions with financial applications is that in many cases the regressor is highly persistent, whereas the dependent variable is close to white noise. For example, stock price returns or exchange rate changes appear to be approximately white noise, whereas predictors like dividend yields or interest rate differentials exhibit persistence behaviour akin to that of a unit root or near unit root autoregressive process.
- As shown by Elliott and Stock (1994,ET), the conventional $t$-statistic in the predictive regression can suffer from severe size distortions in such cases.

# Introduction

- Consider testing $H_0 : \beta = 0$ (i.e. $y_t$ unpredictable by $x_{t-1}$) in the predictive regression

$$y_t = \alpha + \beta x_{t-1} + \epsilon_t$$

  where $y_t$ is local-to-white noise (e.g. returns) and $x_t$ is local-to-unit root (e.g. dividend yield).

- A number of papers have focused on developing asymptotically valid tests of this hypothesis, allowing for an unknown local-to-unity parameter in $x_t$ and unknown correlation between $\epsilon_t$ and the innovations to $x_t$ process, e.g.:
  - Cavanagh *et al.* (1995,ET) (Bonferroni bounds that yield conservative tests)
  - Campbell and Yogo (2006,JFE) (point optimal $t$-test and employing confidence belts)
  - Breitung and Demetrescu (2015,JoE) (variable addition and IV methods).

# Introduction

- However, suppose the true DGP is

$$y_t = \alpha + \delta z_{t-1} + \epsilon_t$$

where $z_t$ is some other local-to-unit root process uncorrelated with $x_t$.

- In this case, testing $H_0 : \beta = 0$ in $y_t = \alpha + \beta x_{t-1} + \epsilon_t$ can result in an asymptotically over-sized test.

- This over-size can be interpreted as a tendency to find a spurious predictor of $y_t$: it is incorrectly concluded that $x_{t-1}$ can be used to predict $y_t$ when in actuality $y_t$ is only predictable by $z_{t-1}$.

- Such spurious predictive regression possibilities were highlighted by Ferson *et al.* (2003a,b,JF,Jnl. Inv. Man.) (using simulation) and Deng (2014,J Fin. Ectrx) (using an asymptotic analysis).

# Introduction

- In this paper we show theoretically the potential for spurious predictive regression to arise in the context of a model where $x_t$ and $z_t$ follow similar but uncorrelated persistent processes (modelled as local-to-unity autoregressions), while modelling the coefficient on $z_{t-1}$ as being local-to-zero.

- We find that spurious rejections in favour of $y_t$ being predicted by $x_{t-1}$ can occur very frequently.

- It is important therefore to be able to identify whether or not the potential predictive regression of $y_t$ on $x_{t-1}$ is mis-specified due to omission of a relevant predictor $z_{t-1}$.

# Introduction

- We propose a test for predictive regression invalidity based on the following:
  - If $y_t = \alpha + \beta x_{t-1} + \epsilon_t$ is the true DGP, the persistent component of $y_t$ is present in the regression of $y_t$ on $x_{t-1}$, and the residuals will be stationary
  - If $y_t = \alpha + \delta z_{t-1} + \epsilon_t$ is the true DGP, the persistent component of $y_t$ is *not* present in the regression of $y_t$ on $x_{t-1}$, and the residuals will be persistent
  - So, any remaining persistence in the residuals from the regression of $y_t$ on $x_{t-1}$ must be due to $z_{t-1}$, signalling invalidity of a predictive regression that employs $x_{t-1}$.
- Our proposed test therefore tests for persistence in the residuals from a regression of $y_t$ on $x_{t-1}$, adapting the co-integration tests of Shin (1994) and Leybourne and McCabe (1994,JBES), which are variants of the stationarity test of Kwiatkowski *et al.* (1992,JoE) test (KPSS).

# Introduction

- A difficulty is that under our null (predictive regression validity), our proposed test has a limit distribution that still depends on the local-to-unity parameter in the process for $x_t$.

- This makes it very difficult to control the size of the test since the local-to-unity parameter cannot be consistently estimated.

- We show that a fixed regressor wild bootstrap procedure (cf. Hansen, 2000,JoE) that conditions on $x_{t-1}$ can be implemented to yield an asymptotically size-controlled testing strategy.

- This procedure is also robust to a wide range of non-stationary error volatility patterns which is potentially important for applications to financial and economic time series.

# The Predictive Regression Model

- The DGP we consider for observed $y_t$ is

$$y_t = \alpha_y + \beta_x x_{t-1} + \beta_z z_{t-1} + \epsilon_{yt}, \qquad t = 1, ..., T$$

where

$$x_t = \alpha_x + s_{x,t}, \qquad z_t = \alpha_z + s_{z,t}, \qquad t = 0, ..., T$$
$$s_{x,t} = \rho_x s_{x,t-1} + \epsilon_{xt}, \qquad s_{z,t} = \rho_z s_{z,t-1} + \epsilon_{zt}, \qquad t = 1, ..., T$$

where $\rho_x := 1 - c_x T^{-1}$ and $\rho_z := 1 - c_z T^{-1}$, with $c_x$, $c_z \geq 0$, so that $x_t$ and $z_t$ are persistent unit root or local to unit root autoregressive processes.

- In order to examine the asymptotic local power of the test procedures, we let $\beta_x := g_x T^{-1}$ and $\beta_z := g_z T^{-1}$, so that when $g_x$ and/or $g_z$ are non-zero, $y_t$ is a persistent, but local-to-noise process.

# The Predictive Regression Model

- The innovation vector $\epsilon_t := [\epsilon_{xt}, \epsilon_{zt}, \epsilon_{yt}]'$ is taken to satisfy the following conditions:

$$\begin{bmatrix} \epsilon_{xt} \\ \epsilon_{zt} \\ \epsilon_{yt} \end{bmatrix} = HD_t e_t$$

  where $e_t$ is a $3 \times 1$ vector m.d.s. with $\sigma_t := E(e_t e_t' | \mathcal{F}_{t-1})$ satisfying $T^{-1} \sum_{t=1}^{T} \sigma_t \xrightarrow{p} E(e_t e_t') = I_3$,

$$H := \begin{bmatrix} h_{11} & 0 & 0 \\ 0 & h_{22} & 0 \\ h_{31} & h_{32} & h_{33} \end{bmatrix}, \qquad D_t := \begin{bmatrix} d_{1t} & 0 & 0 \\ 0 & d_{2t} & 0 \\ 0 & 0 & d_{3t} \end{bmatrix}$$

  with $HH'$ strictly positive definite and the $d_{it}$ satisfying $d_{it} := d_i(t/T)$, $i = 1, 2, 3$, with $d_i(\cdot)$ non-stochastic.

- The structure of $H$ imposes zero correlation between $\epsilon_{xt}$ and $\epsilon_{zt}$, while $\epsilon_{yt}$ can be correlated with $\epsilon_{xt}$ and/or $\epsilon_{zt}$.

# The Predictive Regression Model

- Stationary conditional heteroskedasticity is permitted.
- Unconditional heteroskedasticity is also permitted via the time-varying matrix $D_t$.
- Assumptions on $D_t$ allow for, e.g. single or multiple variance or covariance shifts, variances which follow a broken trend, smooth transition variance shifts, etc.
- In the unconditionally homoskedastic case, it is useful to let $D_t = I$ (w.l.o.g.) and define the innovation variance-covariance matrix as

$$HH' =: \Omega := \begin{bmatrix} \sigma_x^2 & 0 & \sigma_{xy} \\ 0 & \sigma_z^2 & \sigma_{zy} \\ \sigma_{xy} & \sigma_{zy} & \sigma_y^2 \end{bmatrix}$$

# The Predictive Regression Model

- Under our assumptions, the following weak convergence result holds:

$$T^{-1/2}\sum_{t=1}^{\lfloor Tr \rfloor}\epsilon_t \xrightarrow{w} \begin{bmatrix} M_{\eta x}(r) \\ M_{\eta z}(r) \\ M_{\eta y}(r) \end{bmatrix} := H \begin{bmatrix} \int_0^r d_1(s)dB_1(s) \\ \int_0^r d_2(s)dB_2(s) \\ \int_0^r d_3(s)dB_3(s) \end{bmatrix}$$

$$= \begin{bmatrix} h_{11}\{\int_0^1 d_1(s)^2\}^{1/2} & 0 & 0 \\ 0 & h_{22}\{\int_0^1 d_2(s)^2\}^{1/2} & 0 \\ h_{31}\{\int_0^1 d_1(s)^2\}^{1/2} & h_{32}\{\int_0^1 d_2(s)^2\}^{1/2} & h_{33}\{\int_0^1 d_3(s)^2\}^{1/2} \end{bmatrix} \begin{bmatrix} B_{\eta 1}(r) \\ B_{\eta 2}(r) \\ B_{\eta 3}(r) \end{bmatrix}$$

with $[B_1(r), B_2(r), B_3(r)]'$ a $3 \times 1$ vector of independent standard Brownian motion processes and

$$B_{\eta i}(r) := \{\int_0^1 d_i(s)^2\}^{-1/2}\int_0^r d_i(s)dB_i(s), \ \ i = 1, 2, 3.$$

- The $B_{\eta i}(r)$ are variance-transformed Brownian motions (Brownian motion under a modification of the time domain).

# The Predictive Regression Model

- Our model allows for a number of possibilities:

$$\begin{array}{lll} H_u: & \beta_x = 0,\ \beta_z = 0 & y_t \text{ unpredictable} \\ H_x: & \beta_x \neq 0,\ \beta_z = 0 & y_t \text{ predictable only by } x_{t-1} \\ H_z: & \beta_x = 0,\ \beta_z \neq 0 & y_t \text{ predictable only by } z_{t-1} \\ H_{xz}: & \beta_x \neq 0,\ \beta_z \neq 0 & y_t \text{ predictable by } x_{t-1} \text{ and } z_{t-1}. \end{array}$$

- Standard predictive regression tests attempt to distinguish between the null of $y_t$ being unpredictable ($H_u$) against an alternative of $y_t$ being predictable only by the observed variable $x_{t-1}$ ($H_x$).
- However, it is possible that $y_t$ is predictable only by the unincluded variable $z_{t-1}$, with $x_{t-1}$ playing no role ($H_z$), in which case any indication of predictability by $x_{t-1}$ would be spurious.
- A final possibility is that $y_t$ is predictable by both $x_{t-1}$ and $z_{t-1}$ ($H_{xz}$); here, a predictive regression model of $y_t$ based on $x_{t-1}$ alone is invalid; indeed, it may not even be possible to estimate a correctly specified predictive regression, eg if $z_t$ is latent.

# The Predictive Regression Model

- We first consider the impact of the presence of $z_{t-1}$ in the DGP on standard predictive regression tests, i.e. we investigate the behaviour of predictive regression tests of $H_u$ against $H_x$ when in fact $H_z$ or $H_{xz}$ is true.

- We then propose a test for possible predictive regression invalidity, where the appropriate composite null is $H_u$ or $H_x$, and the alternative $H_z$ or $H_{zx}$.

# Asymptotic Behaviour of Predictive Regression Tests

- We first consider the basic predictive regression test based on the $t$-ratio for testing $\beta_x = 0$ in the fitted linear regression

$$y_t = \hat{\alpha}_y + \hat{\beta}_x x_{t-1} + \hat{\epsilon}_{yt}, \qquad t = 1, ..., T.$$

- The test statistic is given by

$$t_u := \frac{\hat{\beta}_x}{\sqrt{s_y^2 / \sum_{t=1}^T (x_{t-1} - \bar{x}_{-1})^2}}$$

where $s_y^2 := (T-2)^{-1} \sum_{t=1}^T \hat{\epsilon}_{yt}^2$ and

$$\hat{\beta}_x := \frac{\sum_{t=1}^T (x_{t-1} - \bar{x}_{-1}) y_t}{\sum_{t=1}^T (x_{t-1} - \bar{x}_{-1})^2}$$

with $\bar{x}_{-1} := T^{-1} \sum_{t=1}^T x_{t-1}$.

# Asymptotic Behaviour of Predictive Regression Tests

- The limit distribution of $t_u$ under our most general hypothesis $H_{xz}$ (which includes all other hypotheses as special cases) is given by

$$t_u \xrightarrow{w} \frac{g_x \int_0^1 \bar{M}_{\eta x,c_x}(r)^2 + g_z \int_0^1 \bar{M}_{\eta x,c_x}(r) M_{\eta z,c_z}(r) + \int_0^1 \bar{M}_{\eta x,c_x}(r) dM_{\eta y}(r)}{\sqrt{\left\{ h_{31}^2 \int_0^1 d_1(r)^2 + h_{32}^2 \int_0^1 d_2(r)^2 + h_{33}^2 \int_0^1 d_3(r)^2 \right\} \int_0^1 \bar{M}_{\eta x,c_x}(r)^2}}$$

where

$$\begin{aligned} \bar{M}_{\eta x,c_x}(r) &:= M_{\eta x,c_x}(r) - \int_0^1 M_{\eta x,c_x}(s) ds \\ \bar{M}_{\eta z,c_z}(r) &:= M_{\eta z,c_z}(r) - \int_0^1 M_{\eta z,c_z}(s) ds \end{aligned}$$

$$\begin{aligned} M_{\eta x,c_x}(r) &:= h_{11} \{ \int_0^1 d_1(s)^2 ds \}^{1/2} B_{\eta 1,c_x}(r) \\ M_{\eta z,c_z}(r) &:= h_{22} \{ \int_0^1 d_2(s)^2 ds \}^{1/2} B_{\eta 2,c_z}(r) \end{aligned}$$

$$\begin{aligned} B_{\eta 1,c_x}(r) &:= \int_0^r e^{-(r-s)c_x} dB_{\eta 1}(s) \\ B_{\eta 2,c_z}(r) &:= \int_0^r e^{-(r-s)c_z} dB_{\eta 2}(s). \end{aligned}$$

# Asymptotic Behaviour of Predictive Regression Tests

- While it is well known from Cavanagh *et al.* (1995) that the limit distribution of $t_u$ under $H_u$ depends on the (unknown) value of $c_x$ whenever $\sigma_{xy} \neq 0$, the limit expression also shows the dependence of $t_u$ on $g_z$ under $H_z$ (where $g_x = 0$ but $g_z \neq 0$).

- Use of asymptotic critical values appropriate for $t_u$ (or the feasible versions from Cavanagh *et al.*, 1995, and Campbell and Yogo, 2006) under $H_u$ will not result in a size-controlled test under $H_z$, raising the possibility of spurious rejections in favour of predictability of $y_t$ by $x_{t-1}$ when $y_t$ is actually predictable by $z_{t-1}$.

- Under $H_{xz}$, where both $g_x \neq 0$ and $g_z \neq 0$, any rejection by $t_u$ cannot uniquely be ascribed to the role of $x_{t-1}$, with the test potentially suggesting a well-specified predictive regression when actually the regression is under-specified due to the omission of $z_t$.

# Asymptotic Behaviour of Predictive Regression Tests

- We also analyze the point optimal variant of the $t_u$ test introduced by Campbell and Yogo (2006), denoted by $Q$.

- The $Q$ statistic and its limit are closely related to $t_u$ and its limit, so we would again anticipate potential asymptotic size distortions for $Q$ under $H_z$.

# Asymptotic Behaviour of Predictive Regression Tests

- We now consider simulations of the asymptotic size of $t_u$ and $Q$ under $H_z$ (where $\beta_x = 0$ but $\beta_z \neq 0$).

- Here we abstract from any role that heteroskedasticity plays by setting $d_i(s) = 1$, $i = 1, 2, 3$.

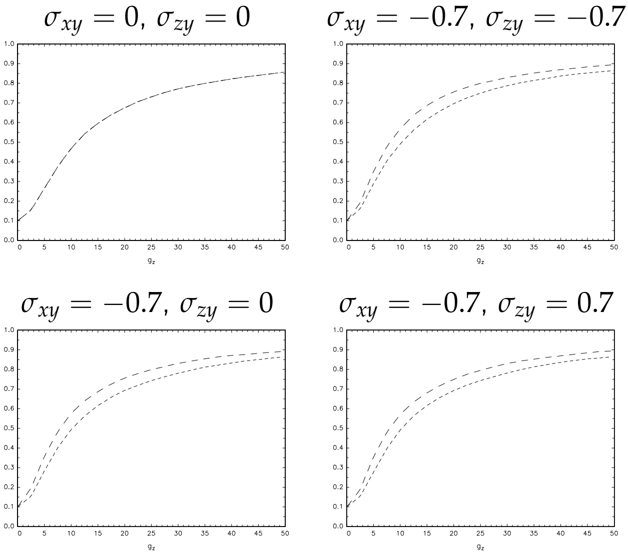
- Critical values are obtained by setting $g_x = g_z = 0$.

- For $t_u$, critical values depend on $c_x$ and $\sigma_{xy}^2 / \sigma_x^2 \sigma_y^2$; for $Q$, critical values depend on $c_x$ alone. These quantities are assumed known, so the results are for infeasible variants of $t_u$ and $Q$.

- We graph nominal $0.10$-level asymptotic sizes of two-sided tests as functions of the parameter $g_z$.

# Asymptotic Size of Predictive Regression Tests

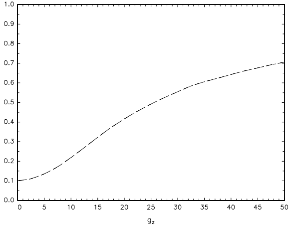
$c_x = c_z = 0$; $t_u$: - - -, $Q$: – –



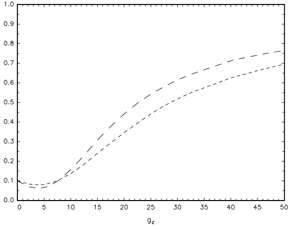
# Asymptotic Size of Predictive Regression Tests

$$c_x = c_z = 5; \;\; t_u\text{: - - -, } Q\text{: – –}$$

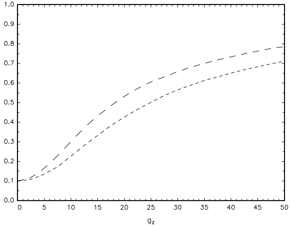
$\sigma_{xy} = 0, \sigma_{zy} = 0$



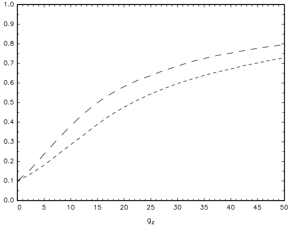

$\sigma_{xy} = -0.7, \sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7, \sigma_{zy} = 0$

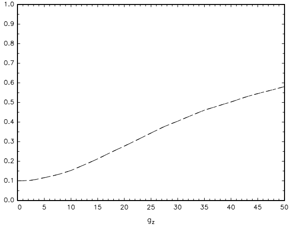



$\sigma_{xy} = -0.7, \sigma_{zy} = 0.7$

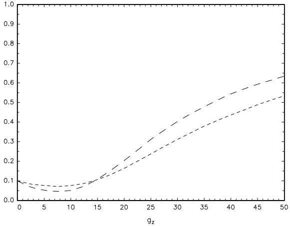


$$c_x = c_z = 10; \;\; t_u\text{: - - -},\; Q\text{: – –}$$

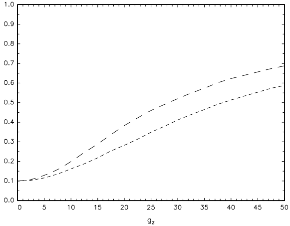
$\sigma_{xy} = 0,\ \sigma_{zy} = 0$



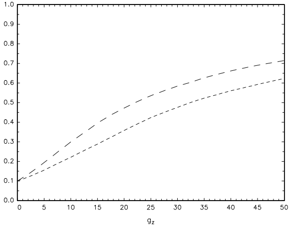

$\sigma_{xy} = -0.7,\ \sigma_{zy} = -0.7$



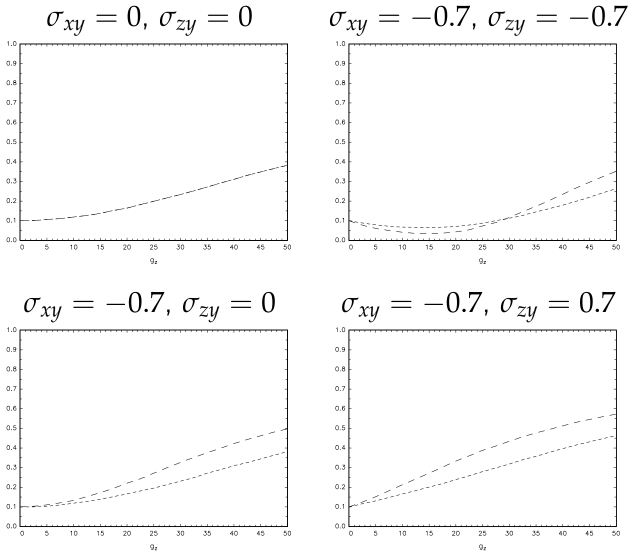

$\sigma_{xy} = -0.7,\ \sigma_{zy} = 0$




$\sigma_{xy} = -0.7,\ \sigma_{zy} = 0.7$



$c_x = c_z = 20$;  $t_u$: - - -, $Q$: – –



$\sigma_{xy} = 0$, $\sigma_{zy} = 0$
$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$
$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$
$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1.0
0 5 10 15 20 25 30 35 40 45 50
$g_z$

# Asymptotic Size of Predictive Regression Tests

- Overall, at least for high-persistence processes, the possibility of finding spurious predictability is an important consideration when employing $t_u$ or $Q$.

- Similar qualitative results will pertain for other predictive regression tests including the recently proposed IV-based tests of Breitung and Demetrescu (2015) whenever a high-persistence IV is used.

- A low-persistence IV test should be less prone to over-size in the presence of a high-persistence omitted predictor $z_{t-1}$, but the price paid for employing such an IV is that when a true predictor $x_{t-1}$ is high-persistence, the IV test will have very poor power.

- Whenever there is scope for high-persistence properties of regressors to yield good power for predictive regression tests, the possibility of spurious predictability is an important issue.

# Stationarity Test for Predictive Regression Invalidity

- Given the potential for standard predictive regression tests to spuriously signal predictability of $y_t$ by $x_{t-1}$ alone when $\beta_z \neq 0$, we now consider a test devised to distinguish between $\beta_z = 0$ and $\beta_z \neq 0$.
- We wish to test the null that $\beta_z = 0$, i.e. $H_u$ or $H_x$, against the alternative that $\beta_z \neq 0$, i.e. $H_z$ or $H_{xz}$.
- Non-rejection would indicate that $z_{t-1}$ plays no role in predicting $y_t$, hence standard predictive regression tests based on $x_{t-1}$ are valid.
- Rejection would indicate the presence of an unincluded $z_{t-1}$ component in the generating process for $y_t$, signalling the invalidity of predictive regression tests based on $x_{t-1}$.

# Stationarity Test for Predictive Regression Invalidity

- Our proposed test is based on testing a null hypothesis of stationarity.

- Consider first the KPSS-type statistic for serially independent errors applied to the residuals $\hat{\epsilon}_{yt}$ from $y_t = \hat{\alpha}_y + \hat{\beta}_x x_{t-1} + \hat{\epsilon}_{yt}$:

$$S := s_y^{-2} T^{-2} \sum_{t=1}^{T} \left( \sum_{i=1}^{t} \hat{\epsilon}_{yi} \right)^2$$

  where $s_y^2 := (T-2)^{-1} \sum_{t=1}^{T} \hat{\epsilon}_{yt}^2$.

- When $\beta_z \neq 0$, the residuals $\hat{\epsilon}_{yt}$ incorporate the omitted $\beta_z z_{t-1}$ term in the generating process for $y_t$, hence the persistence in $z_{t-1}$ is passed to $\hat{\epsilon}_{yt}$.

- A test of $\beta_z = 0$ against $\beta_z \neq 0$ can then be formed as a test for stationarity of $\hat{\epsilon}_{yt}$, rejecting for large values of $S$.

# Stationarity Test for Predictive Regression Invalidity

- To account for the possibility of correlation between $\epsilon_{xt}$ and $\epsilon_{yt}$ ($h_{31} \neq 0$), we follow Shin (1994,ET) by including an additional regressor $\Delta x_t$ in the regression used to construct the KPSS-type statistic.

- We therefore use the fitted linear regression

$$y_t = \hat{\alpha}_y + \hat{\beta}_x x_{t-1} + \hat{\beta}_{\Delta x} \Delta x_t + \hat{e}_t, \qquad t = 1, ..., T$$

and construct $S$ using the residuals $\hat{e}_t$, redefining $S$ as

$$S := s^{-2} T^{-2} \sum_{t=1}^{T} \left( \sum_{i=1}^{t} \hat{e}_i \right)^2$$

where $s^2 := (T-3)^{-1} \sum_{t=1}^{T} \hat{e}_t^2$.

# Stationarity Test for Predictive Regression Invalidity

- Note that $S$ should properly be viewed as a mis-specification test for the putative predictive regression.

- A rejection by this test indicates that the predictive regression based on $x_{t-1}$ alone is invalid, but does not necessarily mean that $x_{t-1}$ is not a valid predictor for $y_t$, as $z_{t-1}$ might be viewed as a proxy for more general mis-specification in the underlying regression model.

- Hence our proposed test is one for the invalidity of the putative predictive regression, not as a test for the invalidity of the putative predictor, $x_{t-1}$.

# Asymptotic Behaviour of Stationarity Test

- The limit distribution of $S$ under $H_{xz}$ is given by

$$S \xrightarrow{w} \{h_{32}^2 \int_0^1 d_2(r)^2 + h_{33}^2 \int_0^1 d_3(r)^2\}^{-1} \int_0^1 \{F(r, c_x) + g_z G(r, c_x, c_z)\}^2$$

where

$$\begin{aligned} F(r, c_x) &:= B_\eta^*(r) - rB_\eta^*(1) - \int_0^r \bar{B}_{\eta 1, c_x} \{\int_0^1 \bar{B}_{\eta 1, c_x}^2\}^{-1} \int_0^1 \bar{B}_{\eta 1, c_x} dB_\eta^* \\ G(r, c_x, c_z) &:= h_{22} \{\int_0^1 d_2^2\}^{1/2} \left[\int_0^r \bar{B}_{\eta 2, c_z} - \int_0^r \bar{B}_{\eta 1, c_x} \{\int_0^1 \bar{B}_{\eta 1, c_x}^2\}^{-1} \int_0^1 \bar{B}_{\eta 1, c_x} B_{\eta 2, c_z}\right] \end{aligned}$$

with

$$\begin{aligned} \bar{B}_{\eta 1, c_x}(r) &:= B_{\eta 1, c_x}(r) - \int_0^1 B_{\eta 1, c_x}(s) \\ \bar{B}_{\eta 2, c_z}(r) &:= B_{\eta 2, c_z}(r) - \int_0^1 B_{\eta 2, c_z}(s) \\ B_\eta^*(r) &:= h_{32} \{\int_0^1 d_2(s)^2\}^{1/2} B_{\eta 2}(r) + h_{33} \{\int_0^1 d_3(s)^2\}^{1/2} B_{\eta 3}(r). \end{aligned}$$

- The presence of $g_z$ in the limit expression is the source of power for $S$ to distinguish between $H_u$ or $H_x$ and $H_z$ or $H_{xz}$ ($S$ is invariant to $\beta_x$).

# Asymptotic Behaviour of Stationarity Test

- Under the null $H_u$ or $H_x$, where $g_z = 0$, the limit can be shown to be invariant to the correlation parameters in $H$.

- However, the limit null distribution of $S$ is not pivotal as it depends on $c_x$ and any unconditional heteroskedasticity in the innovations.

- To account for the dependence of the limit distribution on the heteroskedasticity, we consider a wild bootstrap procedure which is based on the residuals $\hat{e}_t$.

- We also need to account for the dependence on $c_x$; this can be done by conditioning on the observed $x := [x_0, x_1, ..., x_T]'$ when implementing the bootstrap procedure; i.e., using a fixed regressor wild bootstrap.

# Fixed Regressor Wild Bootstrap Stationarity Test

- A conventional approach to obtaining wild bootstrap critical values for $S$ would involve repeated generation of bootstrap samples $y_{t,b}$ that mimic the null behaviour of $y_t$, together with repeated generation of bootstrap samples $x_{t,b}$ that mimic the behaviour of $x_t$.

- Generation of $y_{t,b}$ with suitable properties is straightforward, using a standard wild bootstrap applied to the residuals $\hat{e}_t$.

- However, finding suitable $x_{t,b}$ is problematic due to the dependence of $x_t$ on $c_x$ which cannot be consistently estimated.

- To avoid this problem, we instead consider a wild bootstrap procedure which conditions on $x := [x_0, x_2, ..., x_T]'$, with each bootstrap statistic $S_b^*$ calculated from the $y_{t,b}$ but with the same observed $x_t$ as was used in the construction of $S$.

# Fixed Regressor Wild Bootstrap Stationarity Test

- Our proposed fixed regressor wild bootstrap test:
  1. Construct the test statistic $S$ using the residuals $\hat{e}_t$ from
  $$y_t = \hat{\alpha}_y + \hat{\beta}_x x_{t-1} + \hat{\beta}_{\Delta x}\Delta x_t + \hat{e}_t, \qquad t = 1, ..., T$$
  2. Construct the wild bootstrap innovations $y_{t,b} = \hat{e}_t w_{t,b}$, where $w_{t,b}$, $t = 1, \ldots, T$, is an $IID\, N(0,1)$ sequence
  3. Calculate the fixed regressor wild bootstrap analogue of $S$,
  $$S_b^* := s_{y,b}^{-2} T^{-2} \sum_{t=1}^{T} \left( \sum_{i=1}^{t} \hat{\epsilon}_{yi,b} \right)^2$$
  where $\hat{\epsilon}_{yt,b}$ are the OLS residuals from the fitted regression
  $$y_{t,b} = \hat{\alpha}_{y,b} + \hat{\beta}_{x,b} x_{t-1} + \hat{\epsilon}_{yt,b}, \qquad t = 1, ..., T.$$
  4. Repeat for $b = 1, 2, ..., B$ and calculate the bootstrap (upper tail) $\alpha$-level empirical critical value $cv_{\alpha,B}$ for $S_b^*$
  5. Reject $H_u / H_x$ in favour of $H_z / H_{xz}$ if $S > cv_{\alpha,B}$

# Fixed Regressor Wild Bootstrap Stationarity Test

- The bootstrap sample $y_{t,b}$ is constructed so as to replicate the pattern of heteroskedasticity present in $\hat{e}_t$ (and originating from $\epsilon_{yt}$); this follows because, conditionally on $\hat{e}_t$, $\hat{\epsilon}_{yt,b}$ is independent over time with mean zero and variance $\hat{e}_t^2$.
- The regression in Step 3 makes the conditioning explicit as each $S_b^*$ statistic uses the same $x_{t-1}$ as the regressor.
- We do not include $\Delta x_t$ as an additional regressor in the Step 3 regression, as the $\hat{e}_t$ used to construct $y_{t,b}$ are already free of any effects of correlation between $\epsilon_{xt}$ and $\epsilon_{yt}$.

# Asymptotic Behaviour of Bootstrap Stationarity Test

- We show that the distribution of $S_b^*$, conditional on the data, weakly converges to the random distribution obtained by conditioning the limit of $S$ on the limit of $x$.
- We further show that under $H_u/H_x$, the distribution of the test statistic $S$, conditional on $x$, also converges weakly to the same random distribution, which then allows us to establish the asymptotic validity of the bootstrap test.
- Establishing these results requires the development of a new conditional joint invariance principle for the original and bootstrap data.
- We require the additional key assumption that $\{[e_{2t}, e_{3t}]\}_{t=-\infty}^{\infty}$ is an m.d.s. also w.r.t. $X \vee F_t$, where $X$ and $F_t$ are the $\sigma$-algebras generated by $\{e_{1t}\}_{t=-\infty}^{\infty}$ and $\{[e_{2s}, e_{3s}]\}_{s=-\infty}^{t}$ and $X \vee F_t$ denotes the smallest $\sigma$-algebra containing both $X$ and $F_t$.

# Asymptotic Behaviour of Bootstrap Stationarity Test

- The limit distributions of $S$, conditional on $x$, and $S_b^*$, conditional on the data, are:

$$S|x \stackrel{w}{\to} \{h_{32}^2 \int_0^1 d_2(r)^2 + h_{33}^2 \int_0^1 d_3(r)^2\}^{-1} \int_0^1 \{F(r, c_x) + g_z G(r, c_x, c_z)\}^2 \Big| B_1$$

$$S_b^*|x, y, z \stackrel{w}{\to} \{h_{32}^2 \int_0^1 d_2(r)^2 + h_{33}^2 \int_0^1 d_3(r)^2\}^{-1} \int_0^1 F^\dagger(r, c_x)^2 \Big| B_1$$

where

$$\begin{aligned} F^\dagger(r, c_x) &:= B_\eta^{\dagger *}(r) - rB_\eta^{\dagger *}(1) - \int_0^r \bar{B}_{\eta 1, c_x} \{\int_0^1 \bar{B}_{\eta 1, c_x}^2\}^{-1} \int_0^1 \bar{B}_{\eta 1, c_x} dB_\eta^{\dagger *}(s) \\ B_\eta^{\dagger *}(r) &:= h_{32} \{\int_0^1 d_2(s)^2\}^{1/2} B_{\eta 2}^\dagger(r) + h_{33} \{\int_0^1 d_3(s)^2\}^{1/2} B_{\eta 3}^\dagger(r) \end{aligned}$$

with the $B_{\eta i}^\dagger(r)$, $i = 1, 2, 3$, defined as $B_{\eta i}(r)$, $i = 1, 2, 3$, but with $B_i^\dagger(r)$ replacing $B_i(r)$, where $[B_1^\dagger(r), B_2^\dagger(r), B_3^\dagger(r)]'$ is a standard trivariate Brownian motion independent of $[B_1(r), B_2(r), B_3(r)]'$.

# Asymptotic Behaviour of Bootstrap Stationarity Test

- The limit results show that the bootstrap statistic $S_b^*$, conditional on the data, and the original statistic $S$, conditional on $x$, share the same asymptotic distribution when $g_z = 0$, i.e. under the null hypothesis $H_u/H_x$.

- We show that, as a result, comparison of the original statistic $S$ with the bootstrap critical value $cv_{\alpha,B}$ results in a procedure that has correct asymptotic (in $T$ and $B$) size under $H_u/H_x$, i.e.

$$\lim_{T,B\to\infty} \Pr\left(S > cv_{\alpha,B}\right) = \alpha$$

  establishing the asymptotic validity of our proposed fixed regressor bootstrap test.

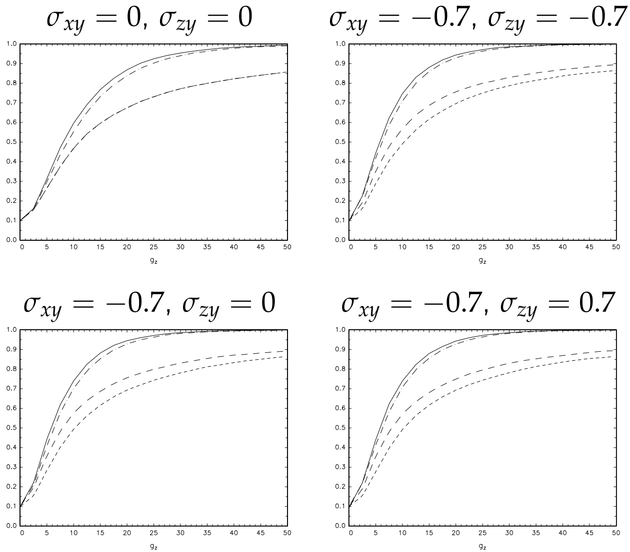
- The source of power of the fixed regressor bootstrap procedure arises from the fact that the limit properties of $cv_{\alpha,B}$ remain unchanged under $H_z/H_{xz}$ while those of $S$ are subject to a stochastic offset dependent on $g_z$.

# Asymptotic Local Power of Stationarity Tests

- We denote the fixed regressor bootstrap procedure, that compares $S$ with $cv_{\alpha,B}$, as $S_B$.
- We now consider simulations of the asymptotic local power of infeasible $S$ (treating $c_x$ as known) and the fixed regressor bootstrap test $S_B$ under $H_z$ (where $\beta_x = 0$ but $\beta_z \neq 0$).
- We graph nominal $0.10$-level local asymptotic powers of the tests as functions of the parameter $g_z$.
- The settings are the same as for the asymptotic size simulations for $t_u$ and $Q$, so the powers of $S_B$ can be compared with the size distortions of $t_u$ and $Q$.

# Asymptotic Local Power of Stationarity Tests

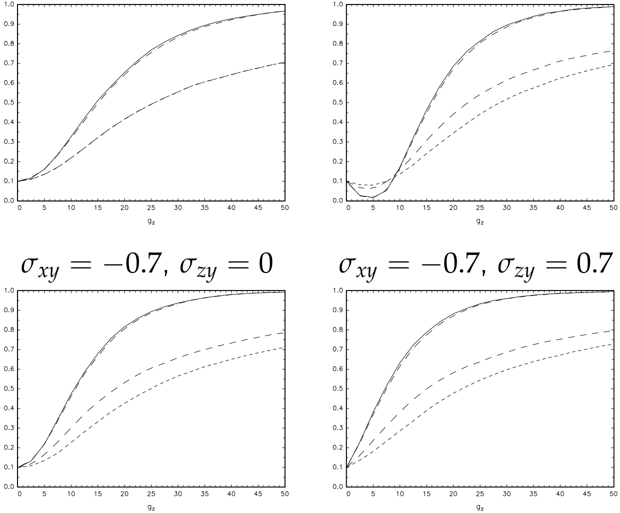
$c_x = c_z = 0$;  $S$: $- \cdot -$, $S_B$: ——,  $t_u$: - - -, $Q$: $- -$



$c_x = c_z = 5$; $S$: – · –, $S_B$: ——, $t_u$: - - -, $Q$: – –

$\sigma_{xy} = 0$, $\sigma_{zy} = 0$ $\quad$ $\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$

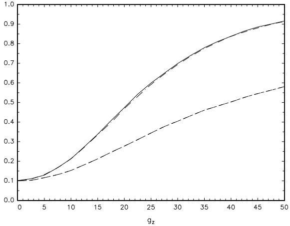
$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$ $\quad$ $\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



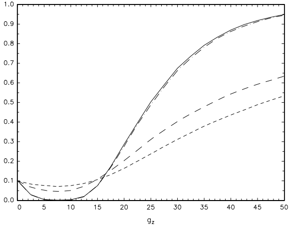
# Asymptotic Local Power of Stationarity Tests

$c_x = c_z = 10$; $S$: $- \cdot -$, $S_B$: ——, $t_u$: - - -, $Q$: – –

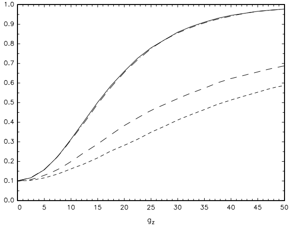
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



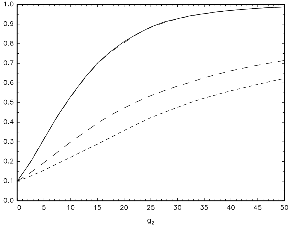

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$

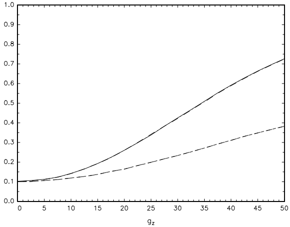



$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$

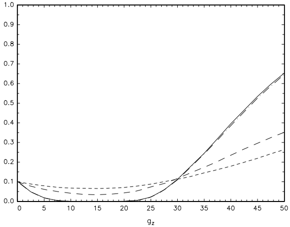


$c_x = c_z = 20$;  $S$: $- \cdot -$, $S_B$: ——,  $t_u$: - - -, $Q$: $- -$

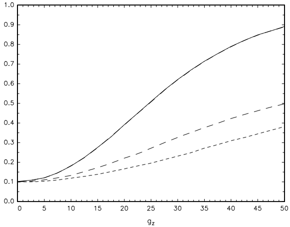
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



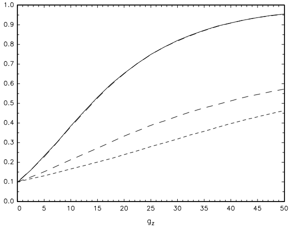

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



# Asymptotic Local Power of Stationarity Tests

- Overall, we see that when the important size problems associated with $t_u$ and $Q$ are apparent, the power of $S_B$ exceeds the size of $t_u$ and $Q$.
- The invalidity of the predictive regression is therefore generally detected with greater frequency than $t_u$ and $Q$ spuriously reject in favour of predictability of $y_t$ by $x_{t-1}$.
- This demonstrates the capability of $S_B$ to act as a meaningful test for predictive regression invalidity.

# Allowing for Additional Serial Correlation

- So far assumed that the $x_t$ innovations $\epsilon_{xt}$ are serially uncorrelated.
- More generally we might consider a linear process assumption for $\epsilon_{xt}$.
- The limiting results continue to hold provided we augment the regression used to calculate $S$ as follows:

$$y_t = \hat{\alpha}_y + \hat{\beta}_x x_{t-1} + \hat{\beta}_{\Delta x} \Delta x_t + \sum_{i=1}^{p} \hat{\delta}_i \Delta x_{t-i} + \hat{e}_t, \qquad t = p+1, ..., T$$

  where $p$ satisfies the standard condition that $1/p + p^3/T \to 0$.

- Note that no such augmentation is needed in the bootstrap regressions used to calculate $S_b^*$.
- Serial correlation in the $z_t$ innovations will have no asymptotic impact under the null $H_u/H_x$; an effect would be seen under $H_z/H_{xz}$.
- As is standard in the predictive regression literature, we maintain the assumption that $\epsilon_{yt}$ is serially uncorrelated.

# Finite Sample Size and Power of Tests

- We now consider finite sample simulations of the size of feasible versions of $t_u$, $Q$, and the preferred IV-based test of Breitung and Demetrescu (2015), denoted $IV_{comb}$, which combines a fractional instrument with a sine function instrument.

- We also consider finite sample simulations of the power of $S_B$ using $B = 499$ bootstrap replications.

- To begin, we focus on the homoskedastic case and report the finite sample analogues of the limit simulations for $H_z$, with $T = 200$.

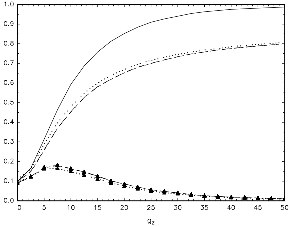
- We also evaluate the performance of a diagnostic procedure, whereby a given predictive regression test ($t_u$, $Q$ or $IV_{comb}$) is only applied if $S_B$ fails to reject.

- This assesses the efficacy of using the $S_B$ test as a diagnostic screen to reduce the degree of predictive regression test over-size.

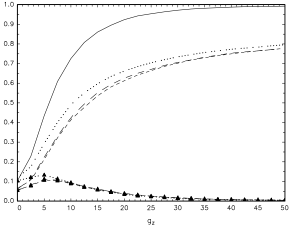
# Finite Sample Size and Power of Tests

$c_x = c_z = 0$; $S_B$: ——, $t_u$: - - -, $Q$: – –, $IV_{comb}$: · · ·, ▲: pre-screened

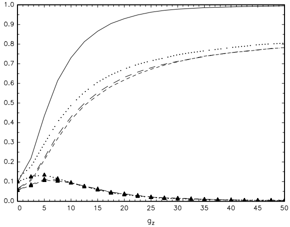
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



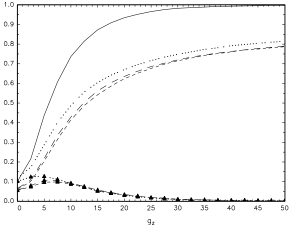

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$



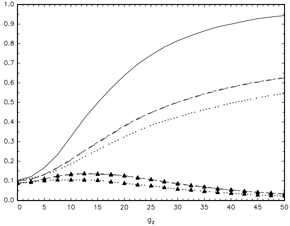

$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



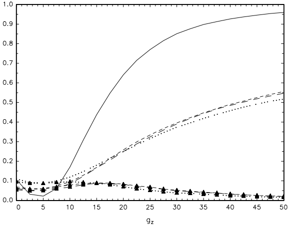
# Finite Sample Size and Power of Tests

$c_x = c_z = 5$; $S_B$: ——, $t_u$: - - -, $Q$: – –, $IV_{comb}$: $\cdots$, ▲: pre-screened

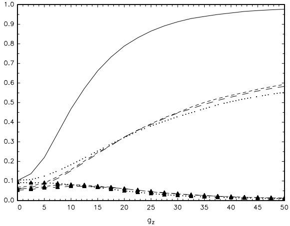
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



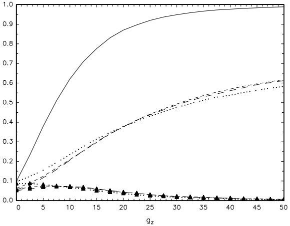

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$



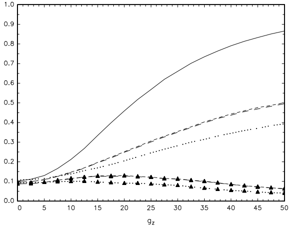

$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



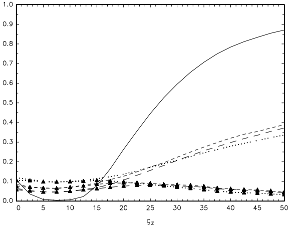
# Finite Sample Size and Power of Tests

$c_x = c_z = 10$; $S_B$: ——, $t_u$: - - -, $Q$: – –, $IV_{comb}$: $\cdot\cdot\cdot$, ▲: pre-screened

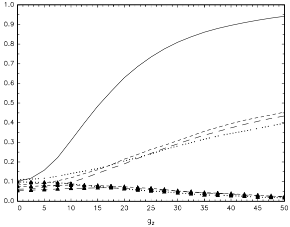
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



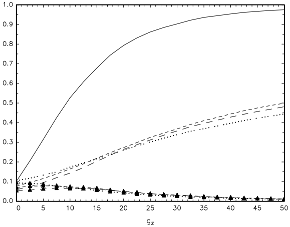

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$



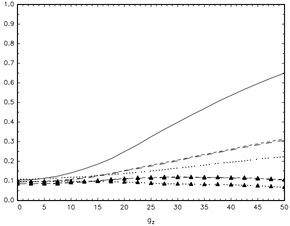

$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



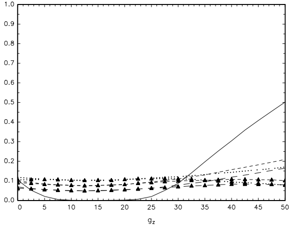
# Finite Sample Size and Power of Tests

$c_x = c_z = 20$;  $S_B$: ——,  $t_u$: - - -, $Q$: – –,  $IV_{comb}$: · · ·, ▲: pre-screened

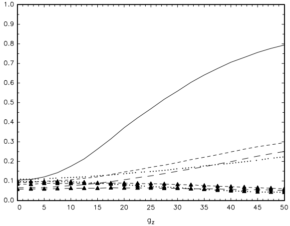
$\sigma_{xy} = 0$, $\sigma_{zy} = 0$



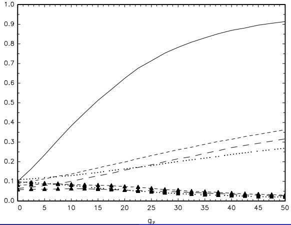

$\sigma_{xy} = -0.7$, $\sigma_{zy} = -0.7$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0$




$\sigma_{xy} = -0.7$, $\sigma_{zy} = 0.7$



# Finite Sample Size and Power of Tests

- The finite sample behaviour of $t_u$, $Q$ and $S_B$ is broadly similar to the asymptotic results.
- The pattern of rejections for $IV_{comb}$ is similar to that for $t_u$ and $Q$.
- The value of $S_B$ is seen in the results for the two stage pre-test-based procedures, with the over-size of $t_u$, $Q$ and $IV_{comb}$ dramatically reduced.
- For the diagnostic screening procedure, rejection frequencies converge to zero as $g_z$ becomes large, driven by the power of $S_B$ increasing in $g_z$.
- Overall, the results suggest a useful role for the predictive regression invalidity test $S_B$.

# Finite Sample Size and Power of Tests

- We also consider the impact of heteroskedasticity in the DGP.

- We simulate the size of $IV_{comb}$ (and its pre-test variant $IV_{comb}^{pre}$), and the size and power of $S_B$, when the error processes are subject to a single break in volatility:

$$d_{it} = \begin{cases} 1 & t \leq \lfloor \tau T \rfloor \\ \sigma_i & t > \lfloor \tau T \rfloor \end{cases}, \qquad i = 1, 2, 3$$

with $\tau = 0.3$ or $\tau = 0.7$ and $\sigma_i = \{1, 4, \frac{1}{4}\}$.

- We consider two cases:
  1. $g_x = g_z = 0$ (size for $IV_{comb}$ and $S_B$)
  2. $g_x = 0, g_z = 25$ (size for $IV_{comb}$, power for $S_B$)

# Finite Sample Size and Power of Tests

$c_x = 0$; $g_x = g_z = 0$: Size for $IV_{comb}$ and $S_B$

| | | $\tau = 0.3$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|
| $\sigma_1$ | $\sigma_3$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ |
| 1 | 1 | 0.098 | 0.109 | 0.094 | 0.098 | 0.109 | 0.094 |
| | 4 | 0.101 | 0.108 | 0.094 | 0.101 | 0.113 | 0.089 |
| | $\frac{1}{4}$ | 0.102 | 0.112 | 0.081 | 0.098 | 0.106 | 0.087 |
| 4 | 1 | 0.100 | 0.110 | 0.093 | 0.102 | 0.112 | 0.096 |
| | 4 | 0.099 | 0.109 | 0.099 | 0.102 | 0.118 | 0.104 |
| | $\frac{1}{4}$ | 0.101 | 0.109 | 0.062 | 0.099 | 0.100 | 0.073 |
| $\frac{1}{4}$ | 1 | 0.102 | 0.112 | 0.093 | 0.099 | 0.111 | 0.094 |
| | 4 | 0.103 | 0.107 | 0.079 | 0.103 | 0.110 | 0.074 |
| | $\frac{1}{4}$ | 0.103 | 0.115 | 0.099 | 0.098 | 0.108 | 0.091 |

# Finite Sample Size and Power of Tests

$c_x = 0$; $g_x = 0$, $g_z = 25$: Size for $IV_{comb}$, power for $S_B$

| | | | $\tau = 0.3$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ |
| 1 | 1 | 1 | 0.910 | 0.714 | 0.049 | 0.910 | 0.714 | 0.049 |
| | | 4 | 0.478 | 0.434 | 0.192 | 0.585 | 0.504 | 0.158 |
| | | $\frac{1}{4}$ | 0.970 | 0.770 | 0.018 | 0.944 | 0.742 | 0.034 |
| | 4 | 1 | 0.997 | 0.845 | 0.002 | 0.977 | 0.767 | 0.013 |
| | | 4 | 0.905 | 0.736 | 0.052 | 0.815 | 0.624 | 0.075 |
| | | $\frac{1}{4}$ | 0.999 | 0.857 | 0.001 | 0.987 | 0.780 | 0.008 |
| | $\frac{1}{4}$ | 1 | 0.656 | 0.553 | 0.129 | 0.864 | 0.703 | 0.068 |
| | | 4 | 0.245 | 0.263 | 0.179 | 0.534 | 0.487 | 0.167 |
| | | $\frac{1}{4}$ | 0.817 | 0.646 | 0.066 | 0.904 | 0.739 | 0.053 |

# Finite Sample Size and Power of Tests

$c_x = 0$; $g_x = 0$, $g_z = 25$: Size for $IV_{comb}$, power for $S_B$

| | | | $\tau = 0.3$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $S_B$ | $IV_{comb}$ | $IV^{pre}_{comb}$ | $S_B$ | $IV_{comb}$ | $IV^{pre}_{comb}$ |
| 4 | 1 | 1 | 0.907 | 0.721 | 0.054 | 0.912 | 0.687 | 0.041 |
| | | 4 | 0.464 | 0.416 | 0.198 | 0.602 | 0.380 | 0.112 |
| | | $\frac{1}{4}$ | 0.971 | 0.801 | 0.019 | 0.942 | 0.769 | 0.036 |
| | 4 | 1 | 0.996 | 0.858 | 0.003 | 0.968 | 0.755 | 0.022 |
| | | 4 | 0.896 | 0.737 | 0.063 | 0.781 | 0.553 | 0.096 |
| | | $\frac{1}{4}$ | 0.999 | 0.872 | 0.001 | 0.978 | 0.783 | 0.016 |
| | $\frac{1}{4}$ | 1 | 0.679 | 0.545 | 0.104 | 0.886 | 0.662 | 0.040 |
| | | 4 | 0.253 | 0.248 | 0.168 | 0.576 | 0.355 | 0.103 |
| | | $\frac{1}{4}$ | 0.826 | 0.689 | 0.062 | 0.919 | 0.757 | 0.036 |

# Finite Sample Size and Power of Tests

$c_x = 0$; $g_x = 0$, $g_z = 25$: Size for $IV_{comb}$, power for $S_B$

| | | | $\tau = 0.3$ | | | $\tau = 0.7$ | | |
|---|---|---|---|---|---|---|---|---|
| $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ | $S_B$ | $IV_{comb}$ | $IV_{comb}^{pre}$ |
| $\frac{1}{4}$ | 1 | 1 | 0.909 | 0.711 | 0.044 | 0.914 | 0.723 | 0.046 |
| | | 4 | 0.494 | 0.504 | 0.186 | 0.584 | 0.569 | 0.173 |
| | | $\frac{1}{4}$ | 0.975 | 0.738 | 0.011 | 0.943 | 0.739 | 0.033 |
| | 4 | 1 | 0.996 | 0.848 | 0.001 | 0.979 | 0.767 | 0.010 |
| | | 4 | 0.920 | 0.774 | 0.036 | 0.824 | 0.665 | 0.069 |
| | | $\frac{1}{4}$ | 0.999 | 0.855 | 0.000 | 0.989 | 0.773 | 0.006 |
| | $\frac{1}{4}$ | 1 | 0.603 | 0.574 | 0.181 | 0.855 | 0.721 | 0.078 |
| | | 4 | 0.214 | 0.326 | 0.223 | 0.515 | 0.554 | 0.201 |
| | | $\frac{1}{4}$ | 0.785 | 0.613 | 0.093 | 0.897 | 0.738 | 0.059 |

# Finite Sample Size and Power of Tests

- The sizes of $S_B$ are well controlled across all the patterns of time-varying volatility of $\epsilon_{xt}$ and $\epsilon_{yt}$.
- When $g_z > 0$, for a given heteroskedasticity setting, the over-size of $IV_{comb}$ and the power of $S_B$ are increasing in $g_z$.
- Heteroskedasticity can have a large influence on the over-size of $IV_{comb}$ and the level of power attainable by $S_B$.
- The diagnostically screened $IV_{comb}^{pre}$ procedure always achieves a reduction in the over-size of $IV_{comb}$.

# Application to US Equity Data

- We now reconsider the results from the empirical analysis investigating the predictability of excess returns using the U.S. equity data in Campbell and Yogo (2006) [CY].
- CY consider four different series of stock returns, dividend-price ratio, and earnings-price ratio. The first is annual S&P 500 index data over the period 1871–2002. The other three series are annual, quarterly, and monthly NYSE/AMEX value-weighted index data (1926–2002). Data descriptions in CY. Data obtainable from `https://sites.google.com/site/motohiroyogo/home/research/`
- CY analyse the time series behaviour of these data and test for predictability in excess returns (relative to an appropriate risk free rate), using as putative predictors for a variety of sample windows: the dividend-price ratio, $d - p$; the earnings-price ratio, $e - p$; the three-month T-bill rate, $r_3$; and a measure of the long-short yield spread, $y - r_1$. As is conventional, excess returns and the predictor variables appear in logs.

# Application to US Equity Data

- CY argue that all of these possible predictors display high persistence with, in most cases, the 95% confidence interval for the largest autoregressive root containing the value unity. Among the series, their results suggest that $r_3$ and $y - r_1$ are the least persistent. Their empirical findings of high persistence in the predictors are echoed by Breitung and Demetrescu (2015) who additionally report large negative estimates of the correlation between the innovations driving the predictors and those driving returns.
- A priori then,with a number of putative predictors under consideration which appear to be highly persistent, bivariate tests of predictability would seem to be at potential risk from the problems identified in this paper.

# Application to US Equity Data

- Table 4 reproduces the bivariate predictive regressions (the predictor involved being identified in the column headed 'Variable') tests from CY.
- Also included is our predictive regression invalidity statistic, $S$, and the KPSS statistic, denoted $KPSS$, for stationarity of the predictor appearing in that regression, and the heteroskedasticity-robust implementation of the $IV_{comb}$ predictive regression test statistic of Breitung and Demetrescu (2015).
- $S$ is implemented using BIC selection for the order of $p$, starting from $p_{\max} = 12$. For KPSS the long run variance estimate is based on the quadratic spectral kernel with automatic bandwidth selection.
- Wild bootstrap $p$-values for $S$ and $KPSS$ are based on $B = 9999$ bootstrap replications.

Table 4. Application to U.S. Equity Indices

| Series | Obs. | Variable | $S$ | $p$-val. | $KPSS$ | $p$-val. | $IV_{comb}$ | $p$-val. | $Q$ |
|---|---|---|---|---|---|---|---|---|---|
| | | | Panel A: S&P 1880-2002, CRSP 1926-2002 | | | | | | |
| S&P 500 | 123 | $d-p$ | 0.358 | 0.057 | 0.669 | 0.043 | 0.187 | 0.426 | NS |
| | | $e-p$ | 1.111 | 0.000 | 0.449 | 0.087 | 1.087 | 0.139 | $*$ |
| Annual | 77 | $d-p$ | 0.081 | 0.658 | 0.572 | 0.077 | 1.383 | 0.083 | $*$ |
| | | $e-p$ | 0.522 | 0.008 | 0.465 | 0.116 | 0.988 | 0.162 | $*$ |
| Quarterly | 305 | $d-p$ | 0.531 | 0.017 | 1.201 | 0.007 | 0.474 | 0.319 | NS |
| | | $e-p$ | 1.302 | 0.000 | 0.889 | 0.026 | 0.624 | 0.267 | $*$ |
| Monthly | 913 | $d-p$ | 1.449 | 0.000 | 2.588 | 0.000 | -0.423 | 0.337 | NS |
| | | $e-p$ | 1.522 | 0.000 | 1.938 | 0.001 | -0.139 | 0.445 | $*$ |
| | | | Panel B: S&P 1880-1994, CRSP 1926-1994 | | | | | | |
| S&P 500 | 115 | $d-p$ | 0.346 | 0.081 | 0.495 | 0.028 | 0.388 | 0.350 | NS |
| | | $e-p$ | 1.207 | 0.000 | 0.251 | 0.146 | 1.600 | 0.054 | $*$ |
| Annual | 69 | $d-p$ | 0.100 | 0.611 | 0.390 | 0.062 | 1.593 | 0.055 | $*$ |
| | | $e-p$ | 0.803 | 0.002 | 0.272 | 0.222 | 1.206 | 0.114 | $*$ |
| Quarterly | 273 | $d-p$ | 0.894 | 0.001 | 0.753 | 0.009 | 0.451 | 0.327 | NS |
| | | $e-p$ | 2.028 | 0.000 | 0.420 | 0.114 | 0.711 | 0.239 | $*$ |
| Monthly | 817 | $d-p$ | 1.626 | 0.000 | 1.473 | 0.000 | -0.598 | 0.276 | NS |
| | | $e-p$ | 2.434 | 0.000 | 0.839 | 0.021 | -0.164 | 0.435 | $*$ |
| | | | Panel C: CRSP 1952-2002 | | | | | | |
| Annual | 51 | $d-p$ | 0.368 | 0.051 | 0.351 | 0.210 | 1.286 | 0.099 | NS |
| | | $e-p$ | 0.058 | 0.675 | 0.244 | 0.270 | 0.979 | 0.163 | NS |
| | | $r_3$ | 0.071 | 0.726 | 0.269 | 0.151 | -1.391 | 0.082 | NS |
| | | $y-r_1$ | 0.085 | 0.657 | 0.626 | 0.014 | 0.472 | 0.381 | NS |
| Quarterly | 204 | $d-p$ | 0.518 | 0.017 | 0.645 | 0.062 | 1.128 | 0.129 | NS |
| | | $e-p$ | 1.511 | 0.000 | 0.550 | 0.064 | 0.764 | 0.223 | NS |
| | | $r_3$ | 0.071 | 0.659 | 0.585 | 0.017 | -2.661 | 0.004 | $*$ |
| | | $y-r_1$ | 0.235 | 0.146 | 0.855 | 0.003 | 0.946 | 0.172 | $*$ |
| Monthly | 612 | $d-p$ | 0.345 | 0.073 | 1.449 | 0.004 | 0.550 | 0.290 | NS |
| | | $e-p$ | 1.729 | 0.000 | 1.264 | 0.004 | 0.363 | 0.358 | NS |
| | | $r_3$ | 0.091 | 0.535 | 1.296 | 0.000 | -3.439 | 0.000 | $*$ |
| | | $y-r_1$ | 0.422 | 0.028 | 1.373 | 0.000 | 1.856 | 0.032 | $*$ |

**Notes:** Returns are for the annual S&P 500 index and the annual, quarterly, and monthly CRSP value-weighted index. The predictor variables are the log dividend-price ratio $d-p$, the log earnings-price ratio $e-p$, the three-month T-bill rate $r_3$, and the long-short yield spread $y-r_1$. In the column headed $Q$, $*$ (NS) indicates those cases where the $Q$ test of Campbell and Yogo (2006) rejects (does not reject) the null hypothesis of no predictability at the 10% level. The columns headed $p$-val. indicate the $p$-values of the tests in the preceding column calculated as detailed in the main text.

# Conclusion

- We have demonstrated that recently proposed tests for predictability have the potential to spuriously signal a valid predictive regression when another persistent variable is present in the underlying data generation process but not included in the predictive regression model.
- We have proposed a diagnostic test for such predictive regression invalidity based on a stationarity testing approach.
- To allow for an unknown degree of persistence in the predictors, and to allow for both conditional and unconditional heteroskedasticity in the data, a fixed regressor wild bootstrap test procedure was proposed and its asymptotic validity established.
- Monte Carlo simulations suggest the proposed methods work well in practice.
- The analysis and proposed test can easily be extended to permit multiple putative and unincluded predictors.